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Distributional Comparisons Using the Gini Inequality Measure

John Creedy*

Abstract

This paper is aimed at undergraduate and graduate economics students, and public sector economists, who are interested in inequality measurement. It examines the use of the Gini inequality measure to compare income distributions. The implicit distributional value judgements are made explicit, via the use of a particular form of Social Welfare Function. Emphasis is given to the interpretation of changes in inequality.

JEL Classification: D331; D63

Keywords: Inequality; Gini Measure; Excess Share; Pivotal Income.

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1 Introduction

The most widely used summary measure of income inequality in official statistics and public debates is the Gini index.[1] This was devised in 1914 by the Italian statistician Corrado Gini (1884-1965).[2] An attractive feature of Gini's index is that it can be related to the famous Lorenz curve: this plots the cumulative proportion of total income against the corresponding proportion of people, after ranking all incomes in ascending order.[3] It is drawn in a square box with sides of unit length: an illustration is shown in Figure 1. The case of complete equality in the diagram is represented by a 45 degree diagonal line: any given proportion of income is obtained by the same proportion of people. Extreme inequality, where only one person has all the income, implies – for a large population – a curve that follows the base and right-hand side of the square box. A intermediate amount of inequality produces a curve lying below the diagonal (the poorest x per cent of people have less than x per cent of total income).

The 'distance' between the actual Lorenz curve and the line of equality therefore provides a convenient visual indication of the extent of inequality. Gini defined his index using an area measure of distance: it is the area between the Lorenz curve and the diagonal line of equality, divided by the area for the case of extreme inequality. As the latter is 1/2 (since the area of the whole box is 1), the Gini is simply twice the former area. The calculation of this area can conveniently be carried out using a range of formulae, as shown below.[4]

The study of inequality measurement was transformed by Atkinson (1970), whose paper changed the subject from the statistical analysis of a measure of dispersion to a central topic in welfare economics.[5] Atkinson based his own eponymous measure on explicitly stated value judgements, assumed to be held by a hypothetical independent

[1] It is used in many contexts, but for convenience the present discussion uses 'income' throughout.

[2] For a review of Gini's contributions to inequality measurement, see Forcina and Georgi (2005).

[3] Max Otto Lorenz (1876-1959) devised the diagram while he was an undergraduate and published it in 1905 while a graduate student. It appears to be his only scientific publication. The device became known as a Lorenz curve following its naming in the text book by King (1912). Lorenz's original curve actually had the proportion of people on the vertical axis.

[4] For some theoretical distributions the Gini can be expressed in terms of the parameters, but for purposes of inequality measurement such expressions are not useful since the distributions provide only approximations to empirical data.

[5] On the interpretation of distributional changes using Atkinson's measure, see the companion paper, Creedy (2021).

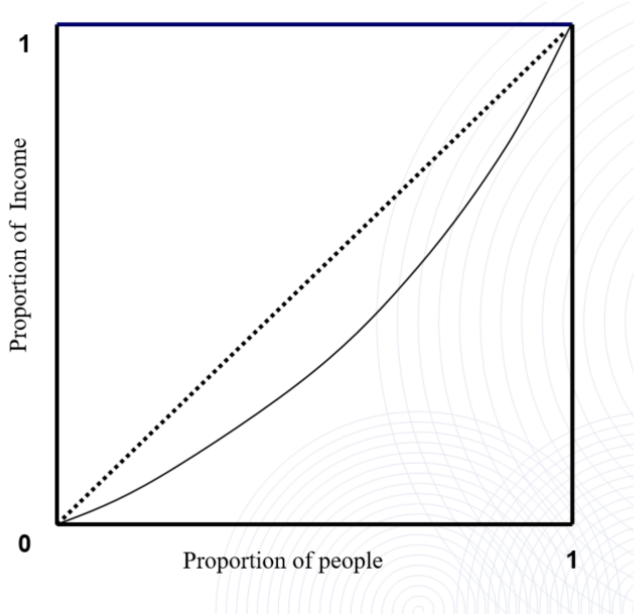

Figure 1: A Lorenz Curve

judge, and capable of being summarised by a social welfare, or evaluation, function. Atkinson measured inequality in terms of the proportional difference between two types of mean income. These are the arithmetic mean income, and the income level, called the 'equally distributed equivalent' income, which, if obtained by everyone, produces the same value of 'social welfare' as the actual distribution. Stimulated by contemporary literature on risk aversion and increasing risk, he used a particular form of welfare function that involves a single parameter, reflecting the 'relative inequality aversion' of the judge, similar to relative risk aversion in the uncertainty context.

Atkinson's paper stimulated a search for the value judgements that are implicit in the Gini index. Although welfare rationales were subsequently developed, and the Gini was extended to include further parameters which aim to capture other aspects of inequality aversion, these are hardly ever mentioned when empirical values of the Gini index are reported in popular debates and official documents. The justification usually given for publishing the Gini measure is that it is 'well known' or 'widely used', rather than any suggestion that the readers are likely to accept the value judgements involved. The present paper therefore attempts to clarify these value judgements, so that people producing and using this almost ubiquitous measure are in a better position to interpret empirical results. In addition, insights are discussed of the ways in which a particular value of the Gini measure, and changes in the measure, can be communicated more widely. It is relatively easy to form a view about a change in an aggregate income measure – a simple sum of values – but it is intuitively much more difficult to appreciate orders of magnitude of an inequality measure which based on combining many individual values.

It is easy to recognise one important value judgement – the 'principle of transfers' – that is implicit in the Gini index. This states that a transfer of income from a richer person to a poorer person, that leaves their positions in the distribution unchanged, represents an 'improvement'. A more technical way of expressing this is to say that the associated social welfare function is concave. In the case of the Gini it is easy to see that this condition holds, even without the welfare function being made explicit. A consideration of the way the Lorenz curve is constructed immediately reveals that such an 'equalising transfer' must move the Lorenz curve, over the relevant ranges, closer to the diagonal line of equality. This principle is regarded in the literature as a basic feature of welfare functions, but it should be recognised that it cannot be expected to be universally approved. For example, consider a three-person case where incomes are 5, 20 and 30 units. A transfer of 5 from the richest person to the middle-income person produces a distribution with incomes of 5, 25 and 25. This obviously shifts the Lorenz curve towards the line of equality, and yet not every independent judge would regard it as an 'improvement', since it leads to increased spread between the low- and high-income ranges of the distribution.

Section 2 begins by introducing several formulae for computing the Gini measure. It also defines an extended form which requires the imposition of a parameter designed to capture a further aspect of inequality aversion. Section 3 explores the value judgements that are implicit in the Gini. The question of how orders of magnitude can be interpreted, or communicated to a wider audience, is considered in Section 4. Section 5 provides a reminder that a particular Gini value can be associated with a range of distributions. Section 6 introduces a closely-related concept, that of the concentration measure, and explains its usefulness in the special context of income taxation, where the redistributive effect is decomposed into separate progressivity and reranking effects. Section 7 concludes.

2 Some Gini Formulae

It was mentioned in the introduction that the Gini is an 'area based' measure. Hence it is first necessary to express the area using a convenient formula.[6] Given a set of incomes for n individuals, suppose values of income, $x_i \geqslant 0$, for $i = 1, ..., n$, are arranged in ascending order, so that $x_1 < x_2 < x_3 < ... < x_n$. One useful way of writing the Gini inequality measure, G, is:

$$G = 1 + \frac{1}{n} - \frac{2}{n^2 \bar{x}} \sum_{i=1}^{n} (n + 1 - i) x_i \tag{1}$$

Here, the weight attached to each income in the summation, $\sum_{i=1}^{n} (n + 1 - i) x_i$, depends on its 'reverse rank' so that, for example, the lowest income, x_1, has a weight of n, while the highest income, x_n, has a weight of 1.[7] If there is extreme inequality, so that only one person has a positive income, x_n, then $\bar{x} = x_n/n$, and (1) becomes $G_{\max} = 1 - 1/n$. For large values of n, this clearly reduces to $G_{\max} = 1$. For complete equality, $x_i = \bar{x}$ for all i, and using the result that $\sum_{i=1}^{n} i = n(n+1)/2$, (1) becomes $G_{\min} = 0$.[8]

Another way of writing G arises from the result that it is half the relative mean difference, and so depends on the absolute differences between all pairs of incomes. Hence:

$$G = \frac{1}{2n^2 \bar{x}} \sum_{j=1}^{n} \sum_{i=1}^{n} |x_i - x_j| \tag{2}$$

The link between (2) and (1) is achieved using the fact that $|x_i - x_j| = x_i - x_j - 2\min(x_i, x_j)$.

A third, and very useful, expression for the Gini is:

$$G = \frac{2}{\bar{x}} Cov(x, F(x)) \tag{3}$$

Here Cov denotes the covariance, and $F(x)$ denotes the proportion of values less than or equal to x. Hence, for the discrete cases considered here, $F(x_i) = i/n$. Using the

[6] Yitzhaki (1993) gives a dozen or so alternative expressions for the Gini, but the following are the most widely used.

[7] The need to arrange incomes in ascending order can slow computer programs down when using very large micro-datasets.

[8] On the calculation of standard errors for the Gini measure, see Giles (2004).

fact that in general, $Cov(y,z) = E(yz) - E(y)E(z)$, where E denotes the expected value, it can be seen that:

$$Cov(x, F(x)) = \frac{1}{n}\sum_{i=1}^{n}\frac{ix_i}{n} - \frac{\bar{x}}{n}\sum_{i=1}^{n}\frac{i}{n} \tag{4}$$

Substituting (4) into (3), and rearranging confirms that it is identical to the expression in (1). The covariance form of the Gini can be convenient in cases where standard statistical software is used. More importantly, it has led to a modified version of the Gini, called the 'extended Gini', $G(v)$, which depends on a parameter, v, and was introduced by Yitzhaki (1983). The modification is given by:

$$G(v) = -\frac{v}{\bar{x}}Cov\left(x, (1 - F(x))^{v-1}\right) \tag{5}$$

Clearly $G(2) = G$, and $G(\infty) = 1 - x_1/\bar{x}$ and, by construction, x_1 is the minimum income.[9] Thus v reflects a form of degree of aversion to inequality.

Finally, suppose weighted values of incomes are available, for example the sample weights provided with survey data to reflect differential responses. This requires a slight modification to the above formulae. If each observation has a weight, w_i, with $\sum_{i=1}^{n} w_i = 1$, the revised form of the cumulative proportion, $F(x_i)$, denoted $\hat{F}(x_i)$, is obtained, using the convention that $w_0 = 0$, as:

$$\hat{F}(x_i) = \frac{w_i}{2} + \sum_{j=0}^{i-1} w_i \tag{6}$$

The weighted Gini measure, G_w, is then given by:

$$G_w = \frac{2}{\bar{x}}\sum_{i=1}^{n} w_i (x_i - \bar{x})\left(\hat{F}(x_i) - \bar{F}\right) \tag{7}$$

where $\bar{x}$ is now the weighted mean $\bar{x} = \sum_{i=1}^{n} w_i x_i$ and $\bar{F}$ is the weighted mean of the $\hat{F}(x_i)$.

3 Gini and Value Judgements

It was mentioned in the introduction that the Gini satisfies the value judgement referred to as the 'Principle of Transfers': this is easily seen from consideration of the Gini as

[9] This is the same as Atkinson's measure for infinite relative inequality aversion.

an area in the Lorenz curve diagram. However, it is useful to know the precise form of the social welfare function that gives rise to the Gini measure, as expressed in the previous section. Several approaches to this problem have been taken. First, it is useful to consider the general nature of the social welfare function involved.

3.1 Social Welfare Functions

The social welfare function is the name given to the evaluation function, W, of an independent judge, and which embodies the value judgements used in assessing distributions. It is a function of all incomes, so that:

$$W = W\left(x_1, x_2, ..., x_n\right) \tag{8}$$

Just as with individual utility functions, a set of social indifference curves can be defined. Consider the simple case where there are just two incomes, so that the associated indifference curves can be drawn in two dimensions. Along indifference curves, total welfare remains constant, and:

$$dW = \frac{\partial W}{\partial x_1} dx_1 + \frac{\partial W}{\partial x_2} dx_2 = 0 \tag{9}$$

Hence the marginal rate of substitution is given by:[10]

$$\left.\frac{dx_1}{dx_2}\right|_W = -\frac{\partial W/\partial x_2}{\partial W/\partial x_1} \tag{10}$$

An indifference curve, drawn without reference to any particular form of W, is shown in Figure 2, as convex and displaying decreasing marginal rates of substitution. Where the welfare function displays anonymity (individuals 1 and 2 have no non-income differences, from the point of view of the judge), the social indifference curves are symmetrical around the upward-sloping 45-degree line.[11]

The solid downward-sloping 45-degree line in the diagram is drawn through the point, C, the location of the two individuals with incomes, x_1 and x_2. Along this line,

[10] The slopes of social indifference curves are often discussed in terms of a thought experiment involving a 'leaky bucket', used in making income transfers. It provides a convenient way of thinking about the nature of a judge's inequality aversion. Of course, the slope is relevant only for small changes.

[11] In the case of Atkinson's inequality measure, the welfare function is homothetic, so that along a ray from the origin (for which relative incomes are constant) the slopes of the social indifference curves are all the same.

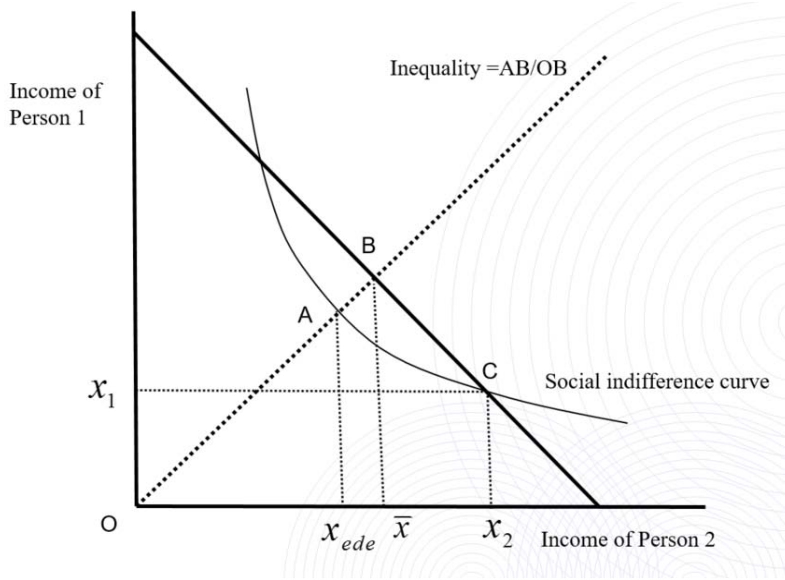

Figure 2: A Social Indifference Curve

total income is constant, so that the intersection of this line with a forward-sloping 45-degree line through the origin gives the arithmetic mean income, $\bar{x}$.

Associated with any W is the concept of an 'equally distributed equivalent' income, x_{ede}, defined as the income which, if obtained by each person, gives the same value of W as the actual distribution. Hence, in this two-person case, x_{ede} is the solution to:

$$W\left(x_{ede}, x_{ede}\right) = W\left(x_1, x_2\right) \tag{11}$$

An aversion to inequality implies that $x_{ede} < \bar{x}$. In Figure 2 this is obtained at the intersection of the indifference curve with the forward sloping 45-degree line, along which incomes are equal. Along the downward-sloping 45-degree line, total income is fixed, whereas along the social indifference curve, total welfare is fixed. The two lines coincide only in the special case where there is no aversion to inequality, in which case the judge cares only about total income.

In general, an inequality measure, I, can be defined in terms of the proportional difference between x_{ede} and $\bar{x}$, or:

$$I = 1 - \frac{x_{ede}}{\bar{x}} \tag{12}$$

In terms of the diagram, this is equal to AB divided by OB. This general approach to defining an inequality measure, and the nature of indifference curves implied by any specified social welfare function, is very useful in thinking about different value judgements. For example, according to a maxi-min approach, where the contribution to the judge's W function, when comparing any two incomes, is equal to the minimum income, means that the indifference curves are L-shaped.

Before moving on to the special case of the Gini measure, one more concept needs to be introduced. Rearranging (12) shows that $x_{ede} = \bar{x}(1 - I)$. Given the way x_{ede} is defined, it is clear that it is a monotonic transformation of W. Hence, an associated form, called the 'abbreviated welfare function', $\widetilde{W}$, is:

$$\widetilde{W} = \bar{x}(1 - I) \tag{13}$$

It is called an abbreviated function because it is expressed in terms of just two variables, the arithmetic mean and the inequality measure, even though both of these are of course functions of all the incomes.[12] It is shown below that the Gini index fits into this class of functions. It is worth stressing here that the Atkinson measure is defined in precisely the same way, and has the same form of abbreviated welfare function, indicating that the judge's willingness to trade income growth for inequality reduction is also the same, except of course that the I measures differ.

3.2 The Gini Social Welfare Function

The nature of the social indifference curves associated with the Gini index is most easily seen by considering the two-person case. Taking the Gini expression in (1) and substituting in (13) gives, after rearranging:

$$\widetilde{W} = \frac{3}{4}x_1 + \frac{1}{4}x_2 \tag{14}$$

Hence the social indifference curves take the particularly simple form:

$$\left.\frac{dx_1}{dx_2}\right|_W = -\frac{1}{3} \tag{15}$$

[12] Some people have taken a somewhat different approach, and have defined a welfare function, for evaluation purposes, in terms of the Gini, but not using the abbreviated form shown above. For example, Kakwani (1980) used the form $\widetilde{W} = \bar{x}/(1+G)$, while Dagum (1990) used $\widetilde{W} = \bar{x}(1-G)/(1+G)$. The implications of these for transfers between pairs of individuals are considered by Creedy and Hurn (1999).

In a diagram corresponding to Figure 2, Gini indifference curves have a kink at the point A, but are otherwise straight lines implying, for example, that if a dollar is taken from person 2, it is necessary only to give 1/3 of a dollar to person 1, to maintain total welfare constant. This applies whatever the initial incomes. Anyone who believes that the amounts involved in making transfers between two individuals should depend on their absolute or relative incomes does not therefore have the same value judgement as those behind the Gini measure. This basic point is rarely, if ever, communicated when Gini measures are reported in popular debates.

The precise result, a slope of $-1/3$ in (15), applies only to the case where there are two people. If, for example, another person is added to the population, with an income in excess of that of the initial two people, then the weights attached to the initial incomes change. In the three-person case, it can be seen that $\widetilde{W}$ is:

$$\widetilde{W} = \frac{5}{9}x_1 + \frac{1}{3}x_2 + \frac{1}{9}x_3 \tag{16}$$

and (15) now becomes $\left.\frac{dx_1}{dx_2}\right|_W = -\frac{3}{5}$, while, for example, $\left.\frac{dx_1}{dx_3}\right|_W = -\frac{1}{5}$. The addition of a richer person means that when a dollar is taken from person 2, it is necessary to give 3/5 of a dollar to person 1. Whereas if the dollar is taken from person 3, it is only required to give 1/5 of a dollar to person 1. Again, these amounts depend only on the ranks of the people concerned, and do not depend on their incomes.

In the general case of n incomes, consider a reduction in j's income of γ with an increase in i's income of θ (where $x_j > x_i$) for which W is unchanged. It can be shown that:

$$\frac{\theta}{\gamma} = \frac{2(n-j)+1}{2(n-i)+1} \tag{17}$$

Hence, the ratio of θ/γ is constant and depends on n and the ranking of the two relevant individuals. For the extended Gini, with the additional parameter, v, it can be shown that:[13]

$$\frac{\theta}{\gamma} = \frac{1+v\left[\left(1-\frac{j}{n}\right)^{v-1} - \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)^{v-1}\right]}{1+v\left[\left(1-\frac{i}{n}\right)^{v-1} - \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)^{v-1}\right]} \tag{18}$$

This kind of comparison provides a useful way to see what is implied by the use of the Gini, but the value judgements involved perhaps still remain less than fully transparent. Further insight is provided as follows.

[13] On comparable results for different Gini-based welfare functions, see Creedy and Hurn (1999).

Sen (1973) showed that a 'pairwise maxi-min' criterion, according to which the welfare level of any pair of individuals is equal to the income of the poorest of the two, gives rise to average welfare across all pairs of $\bar{x}(1-G)$, which, as shown above, is the abbreviated welfare function associated with the Gini measure. Later, Muliere and Scarsini (1989) showed that a similar rationale can be provided for the use of an abbreviated welfare function of the form, $W = \bar{x}(1 - G(v))$, which applies to the extended Gini in (5). Using a simple extension of Sen's maxi-min criterion, they showed that if the welfare of any v-tuple of individuals is equal to the income of the poorest person, then the average of welfare of all v-tuples is $\bar{x}(1 - G(v))$.[14]

However, a rather different, and more transparent approach to deriving the Gini measure from basic value judgements, is to return to the concept of the type of inequality measure defined in equation (12). Consider a welfare function, W, that is written in terms of the per-capita weighted sum:

$$W = \frac{1}{n} \sum_{i=1}^{n} (n+1-i)\, x_i \tag{19}$$

The value judgement is that each income's contribution to social welfare is given a weight depending on the 'reverse-rank' position of that person in the distribution. In this case, the equally distributed equivalent income is given by:

$$x_{ede} = \frac{\sum_{i=1}^{n} (n+1-i)\, x_i}{\sum_{i=1}^{n} (n+1-i)} \tag{20}$$

Substituting this result into (12) gives the associated Gini as:

$$G = 1 - \frac{x_{ede}}{\bar{x}} \tag{21}$$

For large values of n, the resulting expression corresponds to that in (1).[15] From (19), it is immediately clear that the top incomes in a large population are given a very small

[14] A different approach was taken by Lambert (1993), who started from the social welfare function expressed as $W = \sum_i U(x_i)$, where the function, $U(x_i)$, denotes the 'contribution to social welfare' of i's income, as seen by the independent judge. Lambert (1993) explored the idea that U is a function of income *and* the individual's rank position. He showed that it gives rise to an abbreviated welfare function of the form, $W_L = \bar{x}(1 - \kappa G)$, for $0 \leq \kappa \leq 1$, where κ is a parameter which combines with the Gini measure to reflect distributional judgments.

[15] Starting instead from the Gini expression in the previous section, gives: $G = \frac{1+n}{n} - \frac{n(n+1)}{n^2}\left(\frac{x_{ede}}{\bar{x}}\right)$, where x_{ede} is the same as above. Hence the expressions give the same values for large n.

weight, compared with incomes at the bottom of the distribution. Hence the Gini has very little sensitivity to high-income gains: anyone concerned largely with top incomes would have little interest in the Gini measure.

4 Interpreting Gini Values

The previous section has shown that the value judgements inherent in the Gini measure can be easily summarised, despite the fact that it originated in terms of an area in a Lorenz curve diagram. The associated welfare function is the sum (per capita) of the reverse-rank-weighted incomes. The social indifference curves (each side of the 45 degree diagonal from the origin) relating to any two incomes are straight lines whose slopes depend only on their ranks in the population. Yet, there remains a challenge, as with any inequality measure, of communicating just what a particular value of G 'means'. It has been established that it must lie between 0 and 1, but is a value of, say, 0.4 in some sense 'large'?

A first indication is obtained by reference to the abbreviated welfare function, $\widetilde{W} = \bar{x}(1-G)$, for which the social indifference curve, in terms of mean income, $\bar{x}$, and 'equality', $1-G$, has a slope given by:

$$\left.\frac{d\bar{x}}{d(1-g)}\right|_{\widetilde{W}} = \frac{\bar{x}}{1-G} \tag{22}$$

This implies that, along the indifference curve, $d\bar{x}/\bar{x} = d(1-G)/(1-G)$, and the judge regards a given percentage increase in arithmetic mean income as equivalent, in welfare terms, to the same percentage increase in inequality. In terms of changes in G, a proportional increase in $(1-G)$ of, say, δ, translates into a proportional reduction in G of $\delta(1-G)/G$. Therefore, if $G = 0.4$, say, the judge regards a 1.5% increase in G as equivalent to a 1% reduction in $\bar{x}$. If measured inequality is higher, at say $G = 0.6$, a smaller increase in G of 0.6% is equivalent to (would be compensated in welfare terms by) the same 1% reduction in $\bar{x}$. The judge is prepared to sacrifice some total income growth for an inequality reduction.

One possible interpretive approach, suggested by Shorrocks (2005), is to consider an alternative artificial distribution that has the same value of the inequality measure as the actual distribution, but consists of only two income levels (or, equivalently, income shares), though there may be more than two individuals in the artificial distribution. Suppose one person has income, x, and the remaining $n-1$ individuals have $(1-x)/(n-1)$ each. Hence total income is 1 and the arithmetic mean is $1/n$. After substituting into (1), and further simplifying, it can be seen that:

$$G = x - \frac{1}{n} \tag{23}$$

Given that equality implies income shares of $1/n$ for each person, the difference, $x-1/n$, can be said to measure the 'excess share' of the richer person. Hence the Gini is a direct measure of the excess share in this particular two-income, n-person, case. However, it is not clear how useful this is as a communication device. Furthermore, it cannot be used to make comparisons in a two-person context, which otherwise would perhaps provide a more convenient example, because, as mentioned above, this form of the Gini index does not have a maximum of 1 for low n: if $n = 2$, the maximum is 0.5.[16]

In order to illustrate the two-person 'cake cutting' exercise, Subramanian (2002) had earlier used a definition of the Gini measure, G_s, that is normalised to the range $0 \leq G_s \leq 1$ for finite sample sizes.[17] In a two-person context, he showed that if the share of the poorer person is denoted, σ, the resulting Gini is $G_s = 1 - 2\sigma$. Given an empirically obtained value of the Gini for a large population, it could be suggested that it implies that the share going to the poorer person in a two-person case with the same Gini measure is equal to $(1 - G_s)/2$. For example if the Gini is 0.5, the equivalent share of the poorest person is 0.25, and the richer person has 75 per cent of the income: the richer person has an income that is three times larger than that of the poorer person. However, the extent to which interpretation with reference to a two-income or two-person analogy is enlightening is a moot question.

5 Inequality-Neutral Distributional Changes

In considering any single summary measure of a large number of individual values, it is clear that the same value of the summary measure is consistent with a wide range of distributions of those individual values. For example, this is obvious in the case of

[16] Consider the extreme two-person distribution $[0, 1]$, for which the arithmetic mean is 1/2. Substitution in (1) gives $G = 0.5$.

[17] The formula is $G_s = \frac{n+1}{n-1} - \frac{2}{n(n-1)\bar{x}} \sum_{i=1}^{n} (n+1-i)\, x_i$. For large n, this gives the same results as the standard Gini used above.

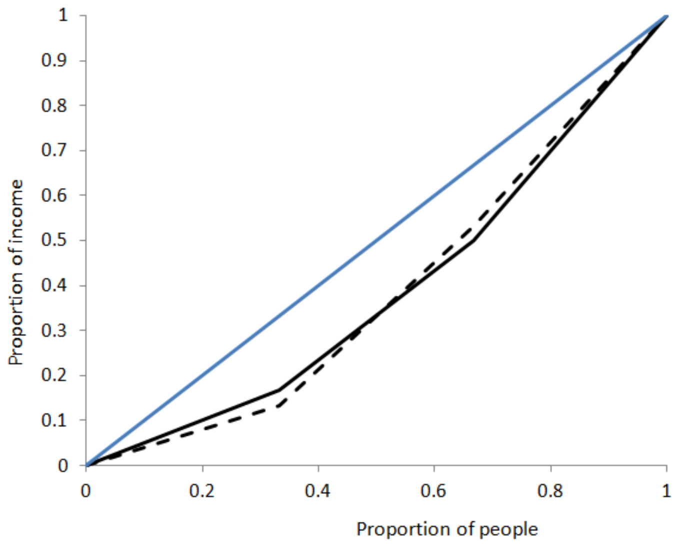

Figure 3: Two Lorenz Curves with Equal Ginis

arithmetic mean income, which depends only on the total income and, of course, the number of individuals. In the present context it is possible to have a mean-preserving change in the distribution of x, for which the value of G remains unchanged.[18] Nevertheless, a judge whose value judgements are characterised by those implicit in the Gini index would be indifferent between those different allocations.

It has been seen that the Gini measure is twice the area between the diagonal line of equality and the Lorenz curve. As an area measure, the Gini is not concerned with precisely which parts of the distribution contribute most to inequality. Two Lorenz curves are shown in Figure 3 for three individuals. The dashed line is closer to the line of equality for the higher part of the distribution but is further from the line of equality for the lower part of the distribution. The two distributions involve simply moving the two Lorenz curve points (corresponding to the cumulative incomes associated with 1/3 and 2/3rds of the population) up or down somewhat. In each Lorenz curve the area contained by the Lorenz curve and the diagonal is the same in each case. The Gini measure cannot distinguish between the two distributions.

Consider instead the distribution $[0, 3, 3]$ for which the Lorenz curve follows the

[18]However, if $n = 2$, there is only one allocation giving the same arithmetic mean and Gini measure. This problem is considered in detail in Creedy (2017) in both Gini and Atkinson contexts.

diagonal beyond the 2/3rds point and the Gini area is all contained to the left of that point (with $G = 0.333$). Another distribution with the same Gini is obtained simply by reducing inequality at the bottom of the distribution and increasing it at the top end. Thus, to give just one alternative, $[0.2, 2.6, 3.2]$ is found to have the same Gini value and of course the same arithmetic mean. A way to view the two distributions is to see that, starting from $[0, 3, 3]$, an equalising transfer of 0.2 is made from person 2 to person 1, and at the same time a disequalising transfer of 0.2 is made from person 2 to person 3. There are two equal transfers from the middle person.

For a judge applying the value judgements behind the Gini measure, the non-uniqueness of the form of the distribution does not matter. However, it cannot be expected that judges with different value judgements, faced with information showing a stable Gini inequality measure over time, or similar Gini values for different demographic groups, will necessarily agree that there are no inequality differences.

6 Gini-Type Concentration Measures

An important context in which the Gini measure plays a substantial role is the comparison between pre-tax incomes, x, and post-tax incomes, y. First, consider inequality of the two distributions. If the tax paid on an income of x is denoted, $T(x)$, then $y = x - T(x)$. In a progressive tax structure, for which by definition the average tax rate increases with income, the inequality of y is less than that of x. The two associated Lorenz curves are shown in Figure 4. Using Gini measures, the redistributive effect of an income tax system, L, can be measured by the difference between the two Gini measures, denoted G_x and G_y, so that:

$$L = G_x - G_y \tag{24}$$

It is often required to produce a measure of the progressivity of the tax structure. The term 'progressivity' refers to the disproportionality of taxes, and is not the same as 'rate progression' which refers to increasing marginal income tax rates with income. Using the values of post-tax income, y, it is possible to produce a curve, similar to a Lorenz curve, in which the individuals are ranked according to their pre-tax incomes. To distinguish it from the Lorenz curve, this is called a concentration curve: an example is shown in Figure 4 as C_y. This is not necessarily the same as the Lorenz curve of

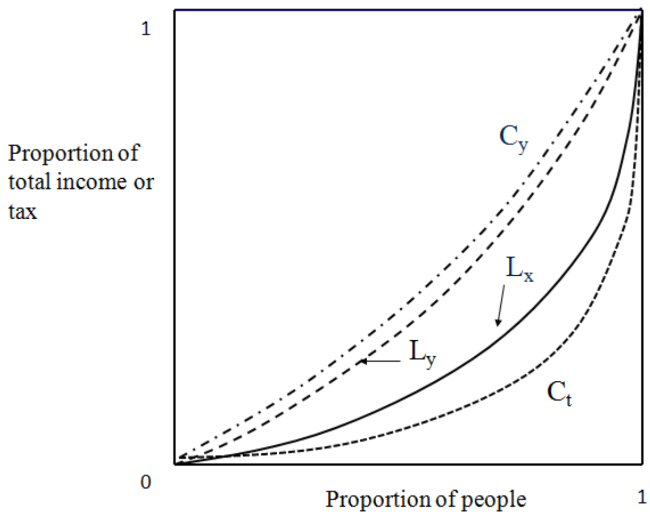

Figure 4: Two Lorenz Curves and Two Concentration Curves

post-tax income because the tax system typically leads to a reranking of individuals when moving from the pre–tax to the post–tax distribution. Reranking arises largely because the tax structure allows various deductions and uses information about some non-income characteristics, such as the existence of dependents.

The concentration curve gives rise to an area measure, the concentration index, that is similar to the Gini inequality measure. The concentration index of net income, C_y, is given simply by substituting y_i for x_i in equation (1). If there is reranking of individuals, then in obtaining the cumulative proportion of total income for the concentration curve, one or more of the values of y must be ahead of where they occur when obtaining the Lorenz curve cumulative proportion. Hence the concentration curve of y must lie inside the Lorenz curve of y over some of its range, as shown in Figure 4.

Similarly, it is possible to plot the proportion of people against the corresponding proportion of total tax paid by those individuals, when the individuals are also ranked in ascending order according to their pre-tax incomes: this gives rise to a tax concentration curve. The associated tax concentration index, C_t, may be obtained by substituting the tax paid, $T(x_i)$, for x_i in (1), remembering to keep the individuals ranked (that is, given appropriate values of i) in the same way as for x_i. An example of a tax concentration curve is shown in Figure 4 as C_t.

If the tax system is proportional, the concentration curve of taxation and the Lorenz curve of pre-tax income coincide, and C_t must be the same as L_x. Furthermore, there cannot be any re–ranking, so that in a proportional system, $L_y = C_y = C_t = L_x$. The curves differ if there is a degree of disproportionality in the tax system. If the tax system is progressive, the average tax rate increases with income, and values of the cumulative tax paid must be less than the corresponding values of cumulative income. Hence, the concentration curve of taxation shows more inequality than the Lorenz curve of pre-tax income, as shown in Figure 4. The difference between the tax concentration index and the Gini measure of x therefore provides a measure of disproportionality or progressivity, K, given by:

$$K = C_t - G_x \tag{25}$$

This is called Kakwani's measure.

The reranking mentioned above, when moving from the distribution of x to that of y, introduces an effect that is contrary to the vertical redistribution intended by the progressive form of the tax function. Without reranking, the concentration curve and the Lorenz curve of y must be identical, in which case G_y and C_y must be equal. Hence the extent of reranking, R, can be measured using:

$$R = G_y - C_y \tag{26}$$

This is called the Atkinson-Plotnick reranking index.

The various measures defined above are all in terms of Gini and closely-related concentration measures. Hence it is not surprising that they can be connected using an explicit formula. First, define the effective total tax ratio, g, as the ratio of the total tax paid to the total pre-tax income of all individuals combined. The relationship between the various measures, due to Kakwani (1984) is:

$$L = G_x - G_y = K\left\{\frac{g}{1-g}\right\} - R \tag{27}$$

Thus the redistributive effect of the tax and transfer system, $G_x - G_y$, is proportional to the Kakwani progressivity measure, K, less the extent of reranking. Importantly, an increase in tax disproportionality need not necessarily reduce the Gini inequality of net income, depending on the tax revenue.

Aronson *et al.* (1994) showed that an extension of this decomposition is possible. They decomposed the redistributive effect of taxation, L, into three components, referred to as the vertical, horizontal, and reranking effects. The vertical effect measures the progressivity of the effective tax schedule, which incorporates no horizontal or reranking effects and is derived from the actual tax schedule by allocating to each individual the average tax paid by the respective pre-tax equals. The horizontal effect relates to the unequal treatment of equals, and reranking captures the presumably unintended treatment of unequals, by the tax system.[19]

7 Conclusions

The Gini inequality measure is ubiquitous in debates concerning inequality. However, the value judgements that are implicit in this measure do not seem to be widely appreciated, and are seldom mentioned when Gini values are reported. This paper has aimed to clarify the nature of these judgements. Reducing a distribution containing a very large number of observations to a single summary measure clearly involves a huge loss of information in an attempt to provide a more transparent indication of the nature and extent of inequality. The trade-off is not worthwhile if users (both producers and consumers) do not have a clear view of precisely what is being measured and how distributional comparisons can be made. The choice of a measure to report when discussing inequality should not be based, as it so often is, on whether it is widely used, but on whether it reflects explicitly stated values. An understanding of these values is important where the role of the economist is seen as investigating the implications of alternative, or at least explicitly stated, value judgements.

[19] The decomposition is based on an extension of the Gini decomposition into population subgroups, with between-group and within-group components, plus a residual. For details and an application, see van de Ven *et al.* (2001).

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