Monadic Bounded Algebras

by

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Abstract

The object of study of the thesis is the notion of monadic bounded algebras (shortly, MBA's). These algebras are motivated by certain natural constructions in free (first-order) monadic logic and are related to free monadic logic in the same way as monadic algebras of P. Halmos to monadic logic (Chapter 1). Although MBA's come from logic, the present work is in algebra. Another important way of approaching MBA's is via bounded graphs, namely, the complex algebra of a bounded graph is an MBA and vice versa.

The main results of Chapter 2 are two representation theorems: 1) every model is a basic MBA and every basic MBA is isomorphic to a model; 2) every MBA is isomorphic to a subdirect product of basic MBA's. As a consequence, every MBA is isomorphic to a subdirect product of models. This result is thought of as an algebraic version of semantical completeness theorem for free monadic logic.

Chapter 3 entirely deals with MBA-varieties. It is proved by the method of filtration that every MBA-variety is generated by its finite special members. Using connections in terms of bounded morphisms among certain bounded graphs, it is shown that every MBA-variety is generated by at most three special (not necessarily finite) MBA's. After that each MBA-variety is equationally characterized.

Chapter 4 considers finitely generated MBA's. We prove that every finitely generated MBA is finite (an upper bound on the number of elements is provided) and that the number of elements of a free MBA on a finite set achieves its upper bound. Lastly, a procedure for constructing a free MBA on any finite set is given.

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Contents

Chapter 1

Introduction

The present work is devoted to the study of monadic bounded algebras which are an algebraic version of free first-order monadic logic. In [7] P. Halmos introduces and studies monadic algebras which are an algebraic version of first-order monadic logic. Following his monadic algebras we develop monadic bounded algebras. The difference between monadic logic and free monadic logic is in the way we treat quantifiers. To be more precise let us give syntax and semantics of free monadic logic.

A language of free monadic logic is a set of symbols arranged as follows:

1. Logical symbols: Parentheses (,); Sentential connective symbols: $\rightarrow$, $\neg$; Variable: x.

2. Parameters: Quantifier symbol: $\exists$; One place predicate symbol: E; Some set of one-place predicate symbols; Some set of constant symbols.

Atomic formulas and well-formed formulas are defined as usual (see e.g. [4, p. 74]).

Suppose we have a language for free monadic logic. Then a structure $\mathfrak{A}$ for the language is a function whose domain is the set of parameters of the language such that

1. $\mathfrak{A}$ assigns to the quantifier $\exists$ a nonempty set $|\mathfrak{A}|$ called the universe (or domain) of $\mathfrak{A}$;

2. $\mathfrak{A}$ assigns to the predicate symbol E a subset $E^{\mathfrak{A}} \subseteq |\mathfrak{A}|$ called the set of actual elements of $\mathfrak{A}$;

3. $\mathfrak{A}$ assigns to each one-place predicate symbol P a subset $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|$;

4. $\mathfrak{A}$ assigns to each constant symbol c an element $c^{\mathfrak{A}} \in |\mathfrak{A}|$.

Note that there is only one variable in the language, and we may call elements in $|\mathfrak{A}| - E^{\mathfrak{A}}$ possible elements. Moreover, we will work with the existential quantifier only.

The satisfaction notion $\mathfrak{A} \models \varphi[a]$, where φ is a formula and $a \in |\mathfrak{A}|$, is defined as usual (see e.g. [4, p. 83-84]) except quantification:

$$\mathfrak{A} \models \exists x\varphi[a] \text{ iff } \mathfrak{A} \models \varphi[b] \text{ for some } b \in E^{\mathfrak{A}}.$$

So, the specifically designated predicate E in the language of free monadic logic singles out a set of actual elements in the domain of a structure for free monadic logic and the range of the existential quantifier is restricted to the set of actual elements. This kind of interpretation is known as "bounded quantification", hence the name of our algebras.

Define $\mathfrak{A} \models \varphi$ (φ is valid in $\mathfrak{A}$) iff $\mathfrak{A} \models \varphi[a]$ for all $a \in |\mathfrak{A}|$.

The following two constructions motivate our study of the main notion of the present work, monadic bounded algebra. In the thesis, $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ designates a two element Boolean algebra.

Let $\mathfrak{A}$ be a structure. Define a (equivalence) relation $\underset{\mathfrak{A}}{\equiv}$ on the set of all formulas by $\varphi \underset{\mathfrak{A}}{\equiv} \psi$ iff $\mathfrak{A} \models \varphi \leftrightarrow \psi$. For a formula φ, define $[\varphi] = \{\psi \mid \varphi \underset{\mathfrak{A}}{\equiv} \psi\}$. Put $B_{\mathfrak{A}} = \{[\varphi] \mid \varphi \text{ is a formula}\}$. For every $[\varphi] \in B_{\mathfrak{A}}$, define a function $\widehat{\varphi} : |\mathfrak{A}| \to \mathbf{2}$ by

$$\widehat{\varphi}(a) = \begin{cases} \mathbf{1}, & \text{if } \mathfrak{A} \models \varphi[a] \\ \mathbf{0}, & \text{if } \mathfrak{A} \not\models \varphi[a] \end{cases}$$

(for every $a \in |\mathfrak{A}|$). Note that:

1. For every $[\varphi] \in B_{\mathfrak{A}}$, $\widehat{\varphi}$ is a well-defined function;
2. Either $\widehat{\exists x\varphi}(a) = \mathbf{1}$ for all $a \in |\mathfrak{A}|$ or $\widehat{\exists x\varphi}(a) = \mathbf{0}$ for all $a \in |\mathfrak{A}|$;
3. Since $\widehat{Ex}(a) = \begin{cases} \mathbf{1}, & \text{if } a \in E^{\mathfrak{A}} \\ \mathbf{0}, & \text{if } a \notin E^{\mathfrak{A}} \end{cases}$, we obtain that $\widehat{Ex}$ is the characteristic function of the subset $E^{\mathfrak{A}} \subseteq |\mathfrak{A}|$.

It will be useful to notice that:

1. $\widehat{\exists x\varphi}(b) = \bigvee_{a \in E^{\mathfrak{A}}} \widehat{\varphi}(a)$, for every $b \in |\mathfrak{A}|$;

   Proof. It is obvious that $\widehat{\varphi}(a) \leq \widehat{\exists x\varphi}(b)$ for every $a \in E^{\mathfrak{A}}$. To be proved that if $\widehat{\varphi}(a) \leq p \in \mathbf{2}$ for every $a \in E^{\mathfrak{A}}$, then $\widehat{\exists x\varphi}(b) \leq p$. Since $p \in \mathbf{2}$, either $p = \mathbf{0}$ or $p = \mathbf{1}$. In the first case we obtain that $\mathfrak{A} \not\models \varphi[a]$ for every $a \in E^{\mathfrak{A}}$, and so $\mathfrak{A} \not\models \exists x\varphi[b]$, i.e. $\widehat{\exists x\varphi}(b) = \mathbf{0} \leq p$. In the second case we obtain $\widehat{\exists x\varphi}(b) \leq p$ just because $p = \mathbf{1}$ is the biggest element in $\mathbf{2}$. □

2. $\bigvee_{a \in E^{\mathfrak{A}}} \widehat{\varphi}(a) = \bigvee_{a \in |\mathfrak{A}|} \left(\widehat{Ex}(a) \wedge \widehat{\varphi}(a)\right)$ (because $\widehat{Ex}$ is the characteristic function of $E^{\mathfrak{A}}$).

Put $\mathbf{M}_0^{\mathfrak{A}} = \{\widehat{\varphi} \mid [\varphi] \in B_{\mathfrak{A}}\}$. So $\mathbf{M}_0^{\mathfrak{A}}$ is a set of functions from the domain $|\mathfrak{A}|$ of the structure $\mathfrak{A}$ to the two-element Boolean algebra $\mathbf{2}$ with the designated (characteristic) function $\widehat{Ex}$ of the subset $E^{\mathfrak{A}} \subseteq |\mathfrak{A}|$.

We are now about to consider the second construction. Suppose D is the set of all constant symbols of the language and $C \subseteq D$. Let S_C be the class of all structures in which the actual elements are just the elements defined by members of C, i.e. $\mathfrak{A} \in S_C$ iff $E^{\mathfrak{A}} = \{c^{\mathfrak{A}} \mid c \in C\}$. Define a (equivalence) relation $\underset{S_C}{\equiv}$ on the set of all formulas by $\varphi \underset{S_C}{\equiv} \psi$ iff $\mathfrak{A} \models \varphi \leftrightarrow \psi$ for all $\mathfrak{A} \in S_C$. For a formula φ, define $[\varphi] = \{\psi \mid \varphi \underset{S_C}{\equiv} \psi\}$. Put $B_{S_C} = \{[\varphi] \mid \varphi \text{ is a formula}\}$. B_{S_C} is a Boolean algebra. For every formula φ, define a function $f(\varphi) : D \to B_{S_C}$ by

$$f(\varphi)(c) = [\varphi(c/x)],$$

where $\varphi(c/x)$ is the formula obtained from φ by replacing the variable x, wherever it occurs free in φ, by the constant symbol c. Note that:

1. For every φ, $f(\varphi)$ is a well-defined function;

2. For $c \in C$, $f(Ex)(c) = \mathbf{1}$ (here $\mathbf{1}$ is the unit element of B_{S_C}).

It will be useful to notice that:

1. $f(\exists x\varphi)(d) = \bigvee_{c\in C} f(\varphi)(c)$ (for every $d \in D$) or, in other words,

$$[\exists x\varphi] = \bigvee\nolimits_{c\in C}[\varphi(c/x)];$$

Proof. Firstly, to be proved that $[\varphi(c/x)] \leq [\exists x\varphi]$ for every $c \in C$, i.e. $\mathfrak{A} \models \varphi(c/x) \to \exists x\varphi$ for all $\mathfrak{A} \in S_C$ and $c \in C$. Suppose $c \in C$ is fixed. Let $\mathfrak{A} \in S_C$, $a \in |\mathfrak{A}|$ and $\mathfrak{A} \models \varphi(c/x)[a]$. Then $\mathfrak{A} \models \varphi[c^{\mathfrak{A}}]$. Since $c \in C$ and $\mathfrak{A} \in S_C$, we have $c^{\mathfrak{A}} \in E^{\mathfrak{A}}$. It follows from $\mathfrak{A} \models \varphi[c^{\mathfrak{A}}]$ and $c^{\mathfrak{A}} \in E^{\mathfrak{A}}$ that $\mathfrak{A} \models \exists x\varphi[a]$. Therefore $\mathfrak{A} \models (\varphi(c/x) \to \exists x\varphi)[a]$. Hence $\mathfrak{A} \models \varphi(c/x) \to \exists x\varphi$. So $\mathfrak{A} \models \varphi(c/x) \to \exists x\varphi$ for all $\mathfrak{A} \in S_C$. Thus $[\varphi(c/x)] \leq [\exists x\varphi]$. Secondly, to be proved that if $[\varphi(c/x)] \leq [\psi]$ for every $c \in C$, then $[\exists x\varphi] \leq [\psi]$, i.e. if $\mathfrak{A} \models \varphi(c/x) \to \psi$ for all $\mathfrak{A} \in S_C$ and $c \in C$, then $\mathfrak{A} \models \exists x\varphi \to \psi$ for all $\mathfrak{A} \in S_C$. Suppose $\mathfrak{A} \in S_C$, $a \in |\mathfrak{A}|$ and $\mathfrak{A} \models \exists x\varphi[a]$. Hence $\mathfrak{A} \models \varphi[b]$ for some $b \in E^{\mathfrak{A}}$. Since $b \in E^{\mathfrak{A}}$ and $\mathfrak{A} \in S_C$, we obtain $b = c_b^{\mathfrak{A}}$ for some $c_b \in C$. Then $\mathfrak{A} \models \varphi(c_b/x)[a]$. Therefore $\mathfrak{A} \models \psi[a]$. So $\mathfrak{A} \models (\exists x\varphi \to \psi)[a]$. Hence $\mathfrak{A} \models \exists x\varphi \to \psi$. Thus $\mathfrak{A} \models \exists x\varphi \to \psi$ for all $\mathfrak{A} \in S_C$. So $[\exists x\varphi] \leq [\psi]$. $\square$

2. $\bigvee_{c\in C} f(\varphi)(c) = \bigvee_{c\in D}(f(Ex)(c) \wedge f(\varphi)(c))$ or, in other words,

$$[\exists x\varphi] = \bigvee\nolimits_{c\in D}[Ec \wedge \varphi(c/x)].$$

Proof. Firstly, to be proved that $[Ec \wedge \varphi(c/x)] \leq [\exists x\varphi]$ for every $c \in D$, i.e. $\mathfrak{A} \models Ec \wedge \varphi(c/x) \to \exists x\varphi$ for all $\mathfrak{A} \in S_C$ and $c \in D$. Suppose $c \in D$ is fixed. Let $\mathfrak{A} \in S$ and $\mathfrak{A} \models Ec \wedge \varphi(c/x)$. Then $\mathfrak{A} \models Ec$ and $\mathfrak{A} \models \varphi(c/x)$. Hence $c^{\mathfrak{A}} \in E^{\mathfrak{A}}$ and $\mathfrak{A} \models \varphi[c^{\mathfrak{A}}]$. So $\mathfrak{A} \models \exists x\varphi$. Secondly, to be proved that if $[Ec \wedge \varphi(c/x)] \leq [\psi]$ for every $c \in D$, then $[\exists x\varphi] \leq [\psi]$, i.e. if $\mathfrak{A} \models Ec \wedge \varphi(c/x) \to \psi$ for every $\mathfrak{A} \in S_C$ and $c \in D$, then $\mathfrak{A} \models \exists x\varphi \to \psi$ for every $\mathfrak{A} \in S_C$. Let $\mathfrak{A} \in S_C$ and $\mathfrak{A} \models \exists x\varphi$. Hence $\mathfrak{A} \models \varphi[a]$ for some $a \in E^{\mathfrak{A}}$. Since $\mathfrak{A} \in S_C$ and $a \in E^{\mathfrak{A}}$, we obtain $a = c_a^{\mathfrak{A}}$ for some $c_a \in C$. Then $\mathfrak{A} \models Ec_a \wedge \varphi(c_a/x)$. Therefore $\mathfrak{A} \models \psi$. Hence $\mathfrak{A} \models \exists x\varphi \to \psi$ for every $\mathfrak{A} \in S_C$. Thus $[\exists x\varphi] \leq [\psi]$. □

Put $\mathrm{M}_1^C = \{f(\varphi) \mid \varphi \text{ is a formula}\}$. So M_1^C is a set of functions from the set of all constant symbols D of the language to the Boolean algebra B_{S_C} with the designated function $f(Ex)$ (which is not a characteristic function in this case).

Thus free monadic logic naturally provides us with constructions $\mathrm{M}_0^{\mathfrak{A}}$ and M_1^C. Generalizing them we get the notion of functional monadic bounded algebra (see Definition 2.1.1). Abstracting from all functional monadic bounded algebras we obtain (abstract) monadic bounded algebras (see Definition 2.2.1). Roughly speaking, a monadic bounded algebra is a Boolean algebra with a designated element E and an unary operation $\exists$, which satisfies six axioms. If we assume that E is the unit element, then we obtain a monadic algebra of P. Halmos. Hence monadic bounded algebras may be considered as a generalization of monadic algebras.

As soon as axioms for monadic bounded algebras are formulated, the question is whether those axioms are an adequate algebraic characterization of functional monadic bounded algebras or not, i.e. whether the variety generated by all functional monadic bounded algebras and the variety of all monadic bounded algebras are equal or not. It suffices to consider the following two questions:

- Is it true that every functional monadic bounded algebra is a monadic bounded algebra? (i.e. do all functional monadic bounded algebras satisfy the axioms?);
- Is it true that every monadic bounded algebra belongs to the variety generated by all functional monadic bounded algebras?

The answer to the first question is given easily (Section 2.1), whereas the second question requires the following result:

> $(*)$ Every monadic bounded algebra is isomorphic to a subdirect product of models,

where a model is by definition a $\mathbf{2}$-valued functional monadic bounded algebra whose designated function E is the characteristic function (in particular, $\mathrm{M}_0^{\mathfrak{A}}$ is a model in this sense). This result is a combination of two representation theorems:

> Every basic monadic bounded algebra is isomorphic to a model (and every model is basic) (Section 2.3)

and

> Every monadic bounded algebra is isomorphic to a subdirect product of basic monadic bounded algebras (Section 2.4).

Basic monadic bounded algebras are related to monadic bounded algebras as P. Halmos' simple monadic algebras to monadic algebras. A monadic bounded algebra $(\mathbf{A}, E, \exists)$ is basic iff the quantifier $\exists$ satisfies the condition

$$\exists p = \begin{cases} \mathbf{1}, & \text{if } p \wedge E \neq \mathbf{0} \\ \mathbf{0}, & \text{if } p \wedge E = \mathbf{0} \end{cases}$$

whereas a monadic algebra $(\mathbf{A}, \exists)$ is simple iff the quantifier $\exists$ satisfies the condition

$$\exists p = \begin{cases} \mathbf{1}, & \text{if } p \neq \mathbf{0} \\ \mathbf{0}, & \text{if } p = \mathbf{0} \end{cases}$$

The result $(*)$ may be thought of as an algebraic version of the semantical completeness theorem for free monadic logic (see p. 34).

There is another way (different from functional) for obtaining monadic bounded algebras. It is based on bounded graphs and it plays a crucial role in the thesis. A triple $\mathcal{F} = (W, R, E)$, where W is a set, $R \subseteq W \times W$ and $E \subseteq W$ (the marked vertices), is called a marked directed graph. It is well-known that the set $\mathcal{P}(W)$ of all subsets of W is a Boolean algebra. Moreover, $E \in \mathcal{P}(W)$ and it is possible to define an operator $\langle R \rangle : \mathcal{P}(W) \to \mathcal{P}(W)$ (see Definition 2.2.8). So the algebra $(\mathcal{P}(W), E, \langle R \rangle)$ is called the complex algebra of $\mathcal{F}$. On the other hand, if $\mathcal{F}$ satisfies certain properties (see Definition 2.2.9), then it is called a bounded graph. The important thing is that the complex algebra of a bounded graph is a monadic bounded algebra (Lemma 2.2.11) and vice versa (Lemma 2.2.12).

Adapting R. Goldblatt's notion of frame morphisms [6] to bounded graphs, we obtain the notion of bounded morphisms for bounded graphs. These bounded morphisms give raise to homomorphisms of the complex algebras of bounded graphs. The second representation theorem above may be stated as follows:

> Every monadic bounded algebra is isomorphic to a subdirect product of subalgebras of complex algebras of some bounded graphs.

From this using the method of filtration due to E.J. Lemmon [8], we can get the next result:

> Every variety of monadic bounded algebras is generated by its finite special members (Section 3.2),

where special monadic bounded algebras are the complex algebras of a vacuous bounded graph or of bounded graphs of Type I or II (see p. 44). Also because of connections (in terms of isomorphisms and homomorphisms) among special monadic bounded algebras, every variety of monadic bounded algebras is generated by at most three (not necessarily finite) special members (Section 3.3). Hence there are only countably many varieties of monadic bounded algebras.

In [9] D. Monk gives explicit equational characterizations for each variety of monadic algebras. There are analogous characterizations for varieties of monadic bounded algebras in Section 3.4, but we get our algebraic expressions by modifying certain formulas from modal logic due to K. Segerberg [10] instead of modifying D. Monk's equations. As a consequence, we obtain the fact that the equational theory of every MBA-variety is finitely based.

Chapter 4 studies finitely generated monadic bounded algebras and it is based on the paper by H. Bass [1]. Let us just state the essential results of the chapter:

- Every monadic bounded algebra generated by $r < \omega$ many elements has at most $2^{3 \cdot 2^r \cdot 2^{2^r - 1}}$ many elements (Section 4.1);
- Every monadic bounded algebra freely generated by $r < \omega$ many elements has exactly $2^{3 \cdot 2^r \cdot 2^{2^r - 1}}$ many elements (Section 4.2);
- Explicit construction of the monadic bounded algebra freely generated by $r < \omega$ many elements is given (actually, this algebra is the complex algebra of a well defined bounded graph) (Section 4.4).

Chapter 2

Monadic bounded algebras

In [7, p. 37] P. Halmos introduces and studies monadic algebras which are an algebraic version of first-order monadic logic. This chapter essentially follows his lines. Let us briefly state what each section of the chapter is about. In Section 2.1, by generalising our motivating structures $\mathrm{M}_0^{\mathfrak{A}}$ and M_1^{C} (see Chapter 1), we obtain the notion of functional monadic bounded algebras. In Section 2.2, by abstracting from functional monadic bounded algebras, we define (abstract) monadic bounded algebras. Moreover, some elementary facts about monadic bounded algebras are provided as well as an important connection between these algebras and bounded graphs. In Section 2.3 we consider specific monadic bounded algebras (namely, basic monadic bounded algebras) and a representation of them. In Section 2.4 we represent a monadic bounded algebra as a subdirect product of basic monadic bounded algebras.

2.1 Functional monadic bounded algebras

In this section, by generalising the set of functions $\mathrm{M}_0^{\mathfrak{A}}$ and M_1^{C} from Chapter 1, we introduce the notion of functional monadic bounded algebras. Several algebraic properties of them are given (and these properties will be the basis of our abstraction from functional monadic bounded algebras to monadic bounded algebras in the next section).

Let $(\mathbf{B}, \wedge, \vee, ', \mathbf{0}, \mathbf{1})$ be a Boolean algebra, X a set and $X_E \subseteq X$.

The set $\mathbf{B}^X$ of all functions from X to $\mathbf{B}$ is a Boolean algebra with respect to the pointwise operations: for $p, q \in \mathbf{B}^X$, the infimum $p \wedge q$, the supremum $p \vee q$ and the complement p' are defined by

$$(p \wedge q)(x) = p(x) \wedge q(x),\ (p \vee q)(x) = p(x) \vee q(x) \text{ and } p'(x) = (p(x))'$$

for each $x \in X$; the zero and the unit of $\mathbf{B}^X$ are the functions that are constantly equal to $\mathbf{0}$ and to $\mathbf{1}$, respectively (here $\mathbf{0}$ and $\mathbf{1}$ are in $\mathbf{B}$).

Definition 2.1.1. *A Boolean subalgebra $\mathbf{A}$ of $\mathbf{B}^X$ with a designated function $E \in \mathbf{B}^X$ is called a **functional monadic bounded algebra** (or $\mathbf{B}$-valued functional monadic bounded algebra with domain (X, X_E) and a designated function E) iff*

1. *for every $x \in X$, $x \in X_E$ implies $E(x) = \mathbf{1}$;*
2. *for every $p \in \mathbf{A}$, both $\bigvee \{p(x) \mid x \in X_E\}$ and $\bigvee \{E(x) \wedge p(x) \mid x \in X\}$ exist in $\mathbf{B}$ and are equal; and*
3. *for every $p \in \mathbf{A}$, the (constant) function $\exists p$, defined by*

   $$\exists p(y) = \bigvee \{p(x) \mid x \in X_E\}\ (y \in X),$$

   belongs to $\mathbf{A}$.

Example 2.1.2. *The sets $\mathbf{M}_0^{\mathfrak{A}}$ and $\mathbf{M}_1^C$ of functions in Chapter 1 are functional monadic bounded algebras, where $X = |\mathfrak{A}|, X_E = E^{\mathfrak{A}}, \mathbf{B} = \mathbf{2}, E = \widehat{Ex}$ and $X = D, X_E = C, \mathbf{B} = B_{S_C}, E = f(Ex)$, respectively.*

Definition 2.1.3. *The operator $\exists$ on a functional monadic bounded algebra is called a **functional existential quantifier**.*

Theorem 2.1.4. *The functional existential quantifier $\exists$ of a functional monadic bounded algebra $\mathbf{A}$ satisfies the following conditions*

1. *$\exists\mathbf{0} = \mathbf{0}$ (here $\mathbf{0}$ is the zero element of $\mathbf{A}$),*

2. $p \wedge E \leq \exists p$,

3. $\exists(p \wedge \exists q) = \exists p \wedge \exists q$,

4. $\exists\exists p = \exists p$,

5. $\exists(p \vee q) = \exists p \vee \exists q$,

6. $\exists(p \wedge E) = \exists p$,

for all $p, q \in \mathbf{A}$.

Proof. The items (1) and (2) immediately follow from Definition 2.1.1.

3. This item is based on the following fact: if $\{p_i\}$ is a family of elements of $\mathbf{B}$ such that $\bigvee_i p_i$ exists, then, for every $q \in \mathbf{B}$, $\bigvee_i(p_i \wedge q)$ exists and is equal to $(\bigvee_i p_i) \wedge q$.

4. Let $x_0 \in X$ be fixed. Since $\exists p$ is a constant function (i.e. $\exists p(x) = \exists p(y)$ for all $x, y \in X$), the set $\{\exists p(x) \mid x \in X_E\}$ is one-element. Then $\bigvee\{\exists p(x) \mid x \in X_E\} = \exists p(x_0)$. But $\exists\exists p(x_0) = \bigvee\{\exists p(x) \mid x \in X_E\}$. Thus $\exists\exists p(x_0) = \exists p(x_0)$. So $\exists\exists p = \exists p$.

5. Let $x_0 \in X$. Since $p, q, p \vee q \in \mathbf{A}$ and using the accociativity of supremums, we obtain that each of the following supremums exists and

$$\begin{aligned}
\exists(p \vee q)(x_0) &= \bigvee\{(p \vee q)(x) \mid x \in X_E\} \\
&= \bigvee\{p(x) \vee q(x) \mid x \in X_E\} \\
&= \bigvee\{p(x), q(x) \mid x \in X_E\} \\
&= \left(\bigvee\{p(x) \mid x \in X_E\}\right) \vee \left(\bigvee\{q(x) \mid x \in X_E\}\right) \\
&= \exists p(x_0) \vee \exists q(x_0) \\
&= (\exists p \vee \exists q)(x_0).
\end{aligned}$$

So $\exists(p \vee q) = \exists p \vee \exists q$.

6. Let $x_0 \in X$. Since $E(x) = \mathbf{1}$ for every $x \in X_E$, we have $\exists(p \wedge E)(x_0) = \bigvee\{(p \wedge E)(x) \mid x \in X_E\} = \bigvee\{p(x) \wedge E(x) \mid x \in X_E\} = \bigvee\{p(x) \mid x \in X_E\} = \exists p(x_0)$. So $\exists(p \wedge E) = \exists p$. □

2.2 Abstract monadic bounded algebras (MBA's)

In this section, by abstracting from functional monadic bounded algebras via Theorem 2.1.4, we introduce the main notion of the present work, (abstract) monadic bounded algebra. Some elementary facts about monadic bounded algebras are proved. Moreover, there is an important connection between monadic bounded algebras and bounded graphs in the section.

Definition 2.2.1. *A* ***monadic bounded algebra*** *(shortly,* ***MBA****) is a triple* $(\mathbf{A}, E, \exists)$, *where* $\mathbf{A}$ *is a Boolean algebra,* $E \in \mathbf{A}$, *and* $\exists$ *is a mapping from* $\boldsymbol{A}$ *to itself such that*

1. $\exists\mathbf{0} = \mathbf{0}$,
2. $p \wedge E \leq \exists p$,
3. $\exists(p \wedge \exists q) = \exists p \wedge \exists q$,
4. $\exists p = \exists\exists p$,
5. $\exists(p \vee q) = \exists p \vee \exists q$,
6. $\exists p = \exists(p \wedge E)$,

for all $p, q \in \mathbf{A}$.

Remark 2.2.2. *Strictly speaking, we should write* $(\mathbf{A}, \wedge, \vee, ', \mathbf{0}, \mathbf{1}, E, \exists)$ *(and sometimes we do) instead of* $(\mathbf{A}, E, \exists)$ *(so, in terms of universal algebra, the type of MBA's is* $(\wedge, \vee, ', \mathbf{0}, \mathbf{1}, E, \exists)$*). To avoid ambiguity, we occasionally emphasize* $\mathbf{0}^{\mathbf{A}}$, $\mathbf{1}^{\mathbf{A}}$, $E^{\mathbf{A}}$ *and* $\exists^{\mathbf{A}}$. *As usual,* $p - q = p \wedge q'$ *and* $p + q = (p - q) \vee (q - p) = (p \wedge q') \vee (q \wedge p')$.

Definition 2.2.3. *An MBA* $(\mathbf{A}, E, \exists)$ *is* ***trivial*** *iff it has only one element.*

Example 2.2.4. *Every functional monadic bounded algebra is a (abstract) monadic bounded algebra (by Theorem 2.1.4).*

Example 2.2.5. *Let* $(\mathbf{M}, \exists)$ *be a monadic algebra [7, p. 40], i.e.* $\mathbf{M}$ *is a Boolean algebra and the quantifier* $\exists : \mathbf{M} \to \mathbf{M}$ *satisfies the following conditions:*

1. $\exists \mathbf{0} = \mathbf{0}$,
2. $p \leq \exists p$,
3. $\exists(p \wedge \exists q) = \exists p \wedge \exists q$,

for every $p, q \in \mathbf{M}$. *Suppose* $E \in \mathbf{M}$ *is any fixed element. Define* $\exists^E : \mathbf{M} \to \mathbf{M}$ *by* $\exists^E p = \exists(E \wedge p)$. *This represents the notion of bounded quantification on p. 2. It is possible to see that* $(\mathbf{M}, E, \exists^E)$ *is an MBA.*

Now suppose $(\mathbf{A}, E, \exists)$ *is an MBA such that* $E = \mathbf{1}$. *Then* $(\mathbf{A}, \exists)$ *is a monadic algebra. Hence MBA's may be considered as a generalization of monadic algebras.*

In the next lemma we sum up some elementary properties of MBA's (cf. [7, p. 41-43]).

Lemma 2.2.6. *Suppose* $(\mathbf{A}, E, \exists)$ *is an MBA and* $p, q \in \mathbf{A}$.

1. $p \in \exists(\mathbf{A})$ *iff* $\exists p = p$.
2. *If* $p \leq \exists q$, *then* $\exists p \leq \exists q$.
3. *If* $p \leq q$, *then* $\exists p \leq \exists q$ *(i.e.* $\exists$ *is monotone).*
4. $\exists(p \wedge E') = \mathbf{0}$.
5. $\exists(E') = \mathbf{0}$.
6. $\exists E = \exists \mathbf{1}$.
7. $E \leq \exists E$.
8. $\exists p \leq \exists E$.
9. $\exists(\exists E)' = \mathbf{0}$.
10. $\exists(\exists p)' \leq (\exists p)'$.

11. $\exists p - \exists q \leq \exists(p-q)$.

12. $\exists p + \exists q \leq \exists(p+q)$.

13. $\exists(p \wedge (\exists q)') = \exists p \wedge (\exists q)'$.

14. $\exists((\exists p)') = \exists E \wedge (\exists p)'$.

Proof. 1. If $p \in \exists(\mathbf{A})$, then $p = \exists p_0$ for some $p_0 \in \mathbf{A}$; and so $\exists p = \exists\exists p_0 = \exists p_0 = p$. If $p = \exists p$, then $p \in \exists(\mathbf{A})$.

2. Since $\exists p = \exists(p \wedge \exists q) = \exists p \wedge \exists q$, we get $\exists p \leq \exists q$.

3. Since $p \leq q$, we get $p \wedge E \leq q \wedge E \leq \exists q$. Then, by previous item, $\exists(p \wedge E) \leq \exists q$. So $\exists p \leq \exists q$ (by Definition 2.2.1(6)).

4. $\exists(p \wedge E') = \exists(p \wedge E' \wedge E)$ [by Definition 2.2.1(6)] $= \exists(p \wedge \mathbf{0}) = \mathbf{0}$ [by Definition 2.2.1(1)]].

5. Put $p = \mathbf{1}$ in the previous item.

6. $\exists E = \exists(\mathbf{1} \wedge E) = \exists\mathbf{1}$ [by Definition 2.2.1(6)]].

7. $E = E \wedge E \leq \exists E$ [by Definition 2.2.1(2)]].

8. Since $p \leq \mathbf{1}$, we have $\exists p \leq \exists\mathbf{1}$ (by item (3)). Hence $\exists p \leq \exists E$ by item (6).

9. $\mathbf{0} = \exists\mathbf{0} = \exists((\exists E)' \wedge \exists E) = \exists(\exists E)' \wedge \exists E$ [by Definition 2.2.1(3)] $= \exists(\exists E)'$ [by item (8)].

10. Since $(\exists p)' \wedge \exists p = \mathbf{0}$, we have that $\mathbf{0} = \exists((\exists p)' \wedge \exists p)$ [by Definition 2.2.1(1)] $= \exists(\exists p)' \wedge \exists p$ [by Definition 2.2.1(3)]. Therefore $\exists(\exists p)' \leq (\exists p)'$.

11. Since $p \vee q = (p-q) \vee q$, it follows by Definition 2.2.1(5) that $\exists p \vee \exists q = \exists(p-q) \vee \exists q$. Forming the infimum of both sides of this equation with $(\exists q)'$, we obtain $\exists p - \exists q = \exists(p-q) - \exists q \leq \exists(p-q)$. So $\exists p - \exists q \leq \exists(p-q)$.

12. $\exists p + \exists q = (\exists p - \exists q) \vee (\exists q - \exists p) \leq \exists(p-q) \vee \exists(q-p)$ [by previous item]$= \exists((p-q) \vee (q-p))$ [by Definition 2.2.1(5)] $= \exists(p+q)$.

13. Part $\leq$. Since $p \wedge (\exists q)' \leq p$ and $p \wedge (\exists q)' \leq (\exists q)'$, we obtain by item (3) that $\exists(p \wedge (\exists q)') \leq \exists p$ and $\exists(p \wedge (\exists q)') \leq \exists(\exists q)'$. Hence $\exists(p \wedge (\exists q)') \leq \exists p \wedge \exists(\exists q)' \leq \exists p \wedge (\exists q)'$ [by item (10)]. Part $\geq$ is proved by $\exists p \wedge (\exists q)' = \exists p - \exists q = \exists p - \exists\exists q \leq \exists(p - \exists q)$ [by item (11)] $= \exists(p \wedge (\exists q)')$.

14. $\exists((\exists p)') = \exists(E \wedge (\exists p)')$ [by Definition 2.2.1(6)] $= \exists E \wedge (\exists p)'$ [by item (13)]. □

Bounded graphs and their complex algebras play a crucial role in the whole work.

Definition 2.2.7. *A triple $\mathcal{F} = (W, R, E)$, where W is a set, $R \subseteq W \times W$ and $E \subseteq W$ (the marked vertices), is called a **marked directed graph**.*

Definition 2.2.8 (cf. [8, p. 192], [6, p. 16])**.** *For a marked directed graph $\mathcal{F} = (W, R, E)$, the **complex algebra** $\mathbf{P}_{\mathcal{F}}$ is $(\mathcal{P}(W), \cap, \cup, -, \mathbf{0}, \mathbf{1}, E, \langle R \rangle)$, where $\cap, \cup$, and $-$ are set-theoretical intersection, union, and complement, respectively, and $\mathbf{0}$ is $\emptyset$ and $\mathbf{1}$ is W and an operator $\langle R \rangle : \mathcal{P}(W) \to \mathcal{P}(W)$ is defined by*

$$\langle R \rangle X = \{x \in W \mid \text{ there is } y \in W \text{ such that } y \in X \text{ and } xRy\}, \tag{2.2.1}$$

for every $X \in \mathcal{P}(W)$. Note that $E \in \mathcal{P}(W)$.

Definition 2.2.9. *A marked directed graph $\mathcal{F} = (W, R, E)$ is a **bounded graph** iff $\mathcal{F}$ satisfies the following four properties:*

1. *R is transitive, i.e. $\forall x, y, z \in W(xRy \& yRz \to xRz)$,*

2. *R is Euclidean, i.e. $\forall x, y, z \in W(xRy \& xRz \to yRz)$,*

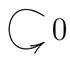
3. *$\forall x, y \in W(xRy \to y \in E)$,*

4. *$\forall x \in W(x \in E \to xRx)$.*

Example 2.2.10.

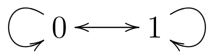
1. *The marked directed graph $\mathcal{F} = (W, R, E)$ is a bounded graph, where $W = \{0\}$, $R = \emptyset$, and $E = \emptyset$. More specifically, $\mathcal{F}$ is a vacuous bounded graph (see Definition 3.1.18).*

2. *The marked directed graph $\mathcal{F} = (W, R, E)$ is a bounded graph, where $W = \{0\}$, $R = W \times W$ and $E = W$. In a picture:*



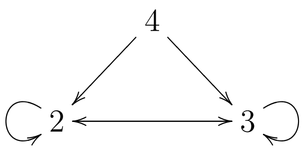
More specifically, $\mathcal{F}$ is a bounded graph of Type I (see Definition 3.1.14).

3. *The marked directed graph $\mathcal{F} = (W, R, E)$ is a bounded graph, where $W = \{0, 1\}$, $R = W \times W$ and $E = W$. In a picture:*



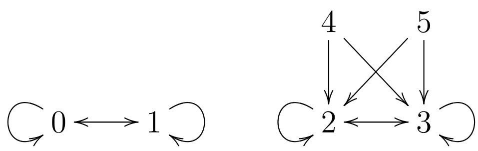
More specifically, $\mathcal{F}$ is a bounded graph of Type I (see Definition 3.1.14).

4. *The marked directed graph $\mathcal{F} = (W, R, E)$ is a bounded graph, where $W = \{2, 3, 4\}$, $R = \{\langle 2, 2\rangle, \langle 3, 3\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$ and $E = \{2, 3\}$. In a picture:*



More specifically, $\mathcal{F}$ is a bounded graph of Type II (see Definition 3.1.16).

5. *The marked directed graph $\mathcal{F} = (W, R, E)$ is a bounded graph, where $W = \{0, 1, 2, 3, 4, 5\}$, $R = \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle, \langle 5, 2\rangle, \langle 5, 3\rangle\}$ and $E = \{0, 1, 2, 3\}$. In a picture:*



Lemma 2.2.11. *Suppose $\mathcal{F} = (W, R, E)$ is a bounded graph. Then the complex algebra $\mathbf{P}_{\mathcal{F}}$ is an MBA.*

Proof. As usual, $(\mathcal{P}(W), \cap, \cup, -, \mathbf{0}, \mathbf{1})$ is a Boolean algebra. By definition, $E \in \mathcal{P}(W)$. It remains to check the six axioms in Definition 2.2.1.

1. From the definition of $\langle R\rangle$, it follows that $\langle R\rangle\emptyset = \emptyset$.

2. To be proved that $X \cap E \subseteq \langle R\rangle X$ (for $X \in \mathcal{P}(W)$). Suppose $x \in X \cap E$. Hence $x \in X$ and $x \in E$. From $x \in E$ follows that xRx. So $x \in \langle R\rangle X$ (since xRx and $x \in X$).

3. To be proved that $\langle R\rangle(X \cap \langle R\rangle Y) = (\langle R\rangle X) \cap (\langle R\rangle Y)$ (for $X, Y \in \mathcal{P}(W)$). For $\subseteq$, suppose $x \in \langle R\rangle(X \cap \langle R\rangle Y)$. Hence there is $y \in (X \cap \langle R\rangle Y)$ such that xRy. Since $y \in (X \cap \langle R\rangle Y)$, we get that $y \in X$ and there is $z \in Y$ with yRz. It follows from xRy and yRz that xRz. Since xRy and $y \in X$, we obtain $x \in \langle R\rangle X$. Since xRz and $z \in Y$, we obtain $x \in \langle R\rangle Y$. Thus $x \in (\langle R\rangle X \cap \langle R\rangle Y)$. For $\supseteq$, suppose $x \in (\langle R\rangle X \cap \langle R\rangle Y)$. Hence xRy, for some $y \in X$, and xRz, for some $z \in Y$. It follows from xRy and xRz that yRz. Since yRz and $z \in Y$, we have $y \in \langle R\rangle Y$. Since $y \in X$ and $y \in \langle R\rangle Y$, we get $y \in X \cap \langle R\rangle Y$. From xRy and $y \in X \cap \langle R\rangle Y$ follows that $x \in \langle R\rangle(X \cap \langle R\rangle Y)$.

4. To be proved that $\langle R\rangle(\langle R\rangle X) = \langle R\rangle X$ (for $X \in \mathcal{P}(W)$). For $\subseteq$, suppose $x \in \langle R\rangle(\langle R\rangle X)$. Hence xRy for some $y \in \langle R\rangle X$. Since $y \in \langle R\rangle X$, we have yRz for some $z \in X$. It follows from xRy and yRz that xRz. Since xRz and $z \in X$, we get $x \in \langle R\rangle X$. For $\supseteq$, suppose $x \in \langle R\rangle X$. Hence xRy for some $y \in X$. Since xRy, we have $y \in E$. Therefore yRy. Since yRy and $y \in X$, we obtain $y \in \langle R\rangle X$. So $x \in \langle R\rangle(\langle R\rangle X)$.

5. To be proved $\langle R\rangle(X \cup Y) = (\langle R\rangle X) \cup (\langle R\rangle Y)$ (for $X, Y \in \mathcal{P}(W)$). For $\subseteq$, suppose $x \in \langle R\rangle(X \cup Y)$. Hence xRy for some $y \in X \cup Y$. Then either xRy, for some $y \in X$, or xRy, for some $y \in Y$. If xRy for $y \in X$, then $x \in \langle R\rangle X$. If xRy for $y \in Y$, then $x \in \langle R\rangle Y$. So either $x \in \langle R\rangle X$ or $x \in \langle R\rangle Y$. Thus $x \in (\langle R\rangle X \cup \langle R\rangle Y)$. For $\supseteq$, suppose $x \in (\langle R\rangle X \cup \langle R\rangle Y)$. Hence either $x \in \langle R\rangle X$ or $x \in \langle R\rangle Y$. Firstly, consider the case $x \in \langle R\rangle X$. Then xRy for some $y \in X$. Therefore xRy for some $y \in X \cup Y$ (since $X \subseteq X \cup Y$). So $x \in \langle R\rangle(X \cup Y)$. Secondly, consider the case $x \in \langle R\rangle Y$. Then xRz for some $z \in Y$. Therefore xRz for some $z \in X \cup Y$ (since $Y \subseteq X \cup Y$). So $x \in \langle R\rangle(X \cup Y)$. Thus in both cases we get $x \in \langle R\rangle(X \cup Y)$.

6. Finally, to be proved that $\langle R\rangle(X \cap E) = \langle R\rangle X$ (for $X \in \mathcal{P}(W)$). For $\subseteq$, suppose $x \in \langle R\rangle(X \cap E)$. Hence xRy for some $y \in X \cap E$. Since $y \in X \cap E$, both $y \in X$ and $y \in E$. So $x \in \langle R\rangle X$ (since xRy and $y \in X$). For $\supseteq$, suppose $x \in \langle R\rangle X$. Hence xRy for some $y \in X$. From xRy follows $y \in E$. Then $y \in X \cap E$. So $x \in \langle R\rangle(X \cap E)$ (since xRy and $y \in X \cap E$).

Thus $\mathbf{P}_{\mathcal{F}}$ is an MBA. □

The converse of the lemma holds as well.

Lemma 2.2.12. *Suppose $\mathcal{F} = (W, R, E)$ is a marked directed graph whose complex algebra $\mathbf{P}_{\mathcal{F}}$ is an MBA. Then $\mathcal{F}$ is a bounded graph.*

Proof. 1. To be proved that R is transitive. Let $x, y, z \in W$, xRy and yRz. Then $x \in \langle R\rangle\{y\}$ and $y \in \langle R\rangle\{z\}$. From $y \in \langle R\rangle\{z\}$ follows that $\{y\} \subseteq \langle R\rangle\{z\}$, and so we have $\langle R\rangle\{y\} \subseteq \langle R\rangle\langle R\rangle\{z\}$ (since $\langle R\rangle$ is monotone). Thus $x \in \langle R\rangle\{y\} \subseteq \langle R\rangle\langle R\rangle\{z\} = \langle R\rangle\{z\}$. Hence xRz.

2. To be proved that R is Euclidean. Let $x, y, z \in W$, xRy and xRz. Then $x \in \langle R\rangle\{y\}$ and $x \in \langle R\rangle\{z\}$, and so $x \in \langle R\rangle\{y\} \cap \langle R\rangle\{z\} = \langle R\rangle(\{y\} \cap \langle R\rangle\{z\})$. Hence xRy' for some $y' \in \{y\} \cap \langle R\rangle\{z\}$. Therefore $y' = y$ and $y' \in \langle R\rangle\{z\}$. So $y \in \langle R\rangle\{z\}$. Thus yRz.

3. To be proved that $\forall x, y \in W(xRy \to y \in E)$. Let $x, y \in W$ and xRy. Then $x \in \langle R\rangle\{y\} = \langle R\rangle(\{y\} \cap E)$. Hence xRy' for some $y' \in \{y\} \cap E$. So $y' = y$ and $y' \in E$. Thus $y \in E$.

4. To be proved that $\forall x \in W(x \in E \to xRx)$. Let $x \in E$. Then $\{x\} \cap E = \{x\}$. Since $\{x\} \cap E \subseteq \langle R\rangle\{x\}$, we have $\{x\} \subseteq \langle R\rangle\{x\}$. Thus xRx. □

2.3 Basic MBA's and their representations as models

This section considers basic MBA's and their representations as models. Firstly, we develop a standard algebraic theory of MBA's (MBA-subalgebras, MBA-ideals, congruences, MBA-homomorphisms, etc.). Secondly, essentially using Stone's representation theorem, we prove our representation theorem. This theorem may be called the first representation theorem because there will be another representation theorem in the next section. Basic MBA's are related to MBA's as P. Halmos' simple monadic algebras to monadic algebras.

Definition 2.3.1. *Let* $(\mathbf{A}, \wedge, \vee, ', \mathbf{0}, \mathbf{1}, E, \exists)$ *be an MBA. A nonempty subset* $\mathbf{A}_0 \subseteq \mathbf{A}$ *is an* ***MBA-subalgebra*** *iff*

- $\mathbf{0}, \mathbf{1}, E \in \mathbf{A}_0$,
- $\mathbf{A}_0$ *is closed under operations* $\wedge, \vee, '$ *and* $\exists$.

Definition 2.3.2. *Suppose* $\mathbf{B}$ *is a Boolean algebra. A subset* $\Delta \subseteq \mathbf{B}$ *is a* ***Boolean ideal*** *iff*

1. *if* $p, q \in \Delta$*, then* $p \vee q \in \Delta$,
2. *if* $p \in \Delta$*, then* $p \wedge q \in \Delta$ *(for every* $q \in \mathbf{B}$*).*

Note this definition of Boolean ideal is equivalent to the usual one in which the second condition is written as follows: if $p \in \Delta$ and $q \leq p$, then $q \in \Delta$.

Now suppose $(\mathbf{A}, E, \exists)$ is an MBA.

Definition 2.3.3. *A subset* $\Delta \subseteq \mathbf{A}$ *is called an* ***MBA-ideal*** *iff both*

1. Δ *is a Boolean ideal in the Boolean algebra* $\mathbf{A}$,
2. $\exists p \in \Delta$ *whenever* $p \in \Delta$.

Definition 2.3.4. *The quantifier* $\exists$ *of* $(\mathbf{A}, E, \exists)$ *is* ***basic*** *iff* $\exists p = \mathbf{1}$ *whenever* $p \wedge E \neq \mathbf{0}$.

In other words, the quantifier $\exists$ is basic iff

$$\exists p = \begin{cases} \mathbf{1}, & \text{if } p \wedge E \neq \mathbf{0} \\ \mathbf{0}, & \text{if } p \wedge E = \mathbf{0} \end{cases}$$

or, equivalently,

$$\exists p = \begin{cases} \mathbf{1}, & \text{if } p \nleq E' \\ \mathbf{0}, & \text{if } p \leq E' \end{cases}.$$

Basicness of MBA's is analogous to simplicity of monadic algebras. To compare them, it suffices just to look at the definitions.

Definition 2.3.5 ([7, p. 41])**.** *The quantifier* $\exists$ *of a monadic algebra* $(\mathbf{A}, \exists)$ *is* ***simple*** *iff*

$$\exists p = \begin{cases} \mathbf{1}, & \text{if } p \neq \mathbf{0} \\ \mathbf{0}, & \text{if } p = \mathbf{0}. \end{cases}$$

Example 2.3.6. *The quantifier of a trivial MBA is basic.*

Example 2.3.7. *The quantifiers of the complex algebras of the bounded graphs in Example 2.2.10(1-4) are basic, whereas the quantifier of the complex algebra of the bounded graph in Example 2.2.10(5) is not.*

Lemma 2.3.8. *If* Δ *is an MBA-ideal in* $(\mathbf{A}, E, \exists)$*, then the relation* $\sim$ *on* $\mathbf{A}$*, defined by*

$$p \sim q \text{ iff } p + q \in \Delta,$$

is a congruence relation on $(\mathbf{A}, E, \exists)$*.*

Proof. Let $p_0 \sim p_1$ and $q_0 \sim q_1$. Hence $p_0 + p_1 \in \Delta$ and $q_0 + q_1 \in \Delta$. Therefore $p_0 \wedge p_1' \in \Delta$, $p_1 \wedge p_0' \in \Delta$, $q_0 \wedge q_1' \in \Delta$ and $q_1 \wedge q_0' \in \Delta$ (since Δ is an MBA-ideal). So $(p_0 \wedge q_0) + (p_1 \wedge q_1) = ((p_0 \wedge q_0) \wedge (p_1 \wedge q_1)') \vee ((p_1 \wedge q_1) \wedge (p_0 \wedge q_0)') = ((p_0 \wedge q_0) \wedge (p_1' \vee q_1')) \vee ((p_1 \wedge q_1) \wedge (p_0' \vee q_0')) = ((p_0 \wedge q_0 \wedge p_1') \vee (p_0 \wedge q_0 \wedge q_1')) \vee ((p_1 \wedge q_1 \wedge p_0') \vee (p_1 \wedge q_1 \wedge q_0')) \in \Delta$. Thus $(p_0 \wedge q_0) \sim (p_1 \wedge q_1)$.

Similarly with $\vee$ and $'$.

Now let $p \sim q$. Then $p + q \in \Delta$. Hence $\exists(p + q) \in \Delta$ (since Δ is an MBA-ideal), and so $\exists p + \exists q \in \Delta$ (by Lemma 2.2.6 (12)). Thus $\exists p \sim \exists q$. □

Lemma 2.3.9. *If* $\sim$ *is a congruence relation on* $(\mathbf{A}, E, \exists)$*, then the set* $\Delta = \{p \in \mathbf{A} \mid \mathbf{0} \sim p\}$ *is an MBA-ideal.*

Proof. Let $p, q \in \Delta$. Then $\mathbf{0} \sim p$ and $\mathbf{0} \sim q$. Hence $\mathbf{0} \vee \mathbf{0} \sim p \vee q$ (since $\sim$ is a congruence relation). Thus $p \vee q \in \Delta$.

Let $p \in \Delta$ and $q \in \mathbf{A}$. Then $\mathbf{0} \sim p$ and $q \sim q$. Hence $\mathbf{0} \wedge q \sim p \wedge q$ (since $\sim$ is a congruence relation). Therefore $\mathbf{0} \sim p \wedge q$. Thus $p \wedge q \in \Delta$.

Let $p \in \Delta$. Then $\mathbf{0} \sim p$, and so $\exists \mathbf{0} \sim \exists p$ (since $\sim$ is a congruence relation). Hence $\mathbf{0} \sim \exists p$. Thus $\exists p \in \Delta$. □

We are going to show that the two constructions in Lemma 2.3.8, 2.3.9 are inverse. Firstly, suppose that Δ is an MBA-ideal in $(\mathbf{A}, E, \exists)$. Let $\sim$ be the congruence relation as defined in Lemma 2.3.8. Then define $\Delta' = \{p \in \mathbf{A} \mid \mathbf{0} \sim p\}$. So $\Delta' = \Delta$, because, for every $p \in \mathbf{A}$, $p \in \Delta'$ iff $\mathbf{0} \sim p$ iff $\mathbf{0} + p \in \Delta$ iff $p \in \Delta$. Secondly, suppose $\sim$ is a congruence relation on $(\mathbf{A}, E, \exists)$. Let Δ be the MBA-ideal as defined in Lemma 2.3.9. Then define $\backsim$ by $p \backsim q$ iff $p + q \in \Delta$. So $\backsim$ and $\sim$ are equal, because, for every $p, q \in \mathbf{A}$, $p \backsim q$ iff $p + q \in \Delta$ iff $\mathbf{0} \sim p + q$ iff $p \sim q$.

Definition 2.3.10. *Suppose* $(\mathbf{A}_1, E^{\mathbf{A}_1}, \exists^{\mathbf{A}_1})$ *and* $(\mathbf{A}_2, E^{\mathbf{A}_2}, \exists^{\mathbf{A}_2})$ *are MBA's. A function* $f : \mathbf{A}_1 \to \mathbf{A}_2$ *is an* ***MBA-homomorphism*** *iff the following conditions hold:*

- $f(p \wedge q) = f(p) \wedge f(q)$,
- $f(p \vee q) = f(p) \vee f(q)$,
- $f(p') = (f(p))'$,
- $f(\mathbf{0}^{\mathbf{A}_1}) = \mathbf{0}^{\mathbf{A}_2}$,
- $f(\mathbf{1}^{\mathbf{A}_1}) = \mathbf{1}^{\mathbf{A}_2}$,
- $f(E^{\mathbf{A}_1}) = E^{\mathbf{A}_2}$,
- $f(\exists^{\mathbf{A}_1} p) = \exists^{\mathbf{A}_2} f(p)$,

for all $p, q \in \mathbf{A}_1$.

If f is one-to-one, then it is called an ***MBA-embedding****. If f is bijective, then it is called an* ***MBA-isomorphism****.*

Definition 2.3.11. *For an MBA-homomorphism*

$$f : (\mathbf{A}_1, E^{\mathbf{A}_1}, \exists^{\mathbf{A}_1}) \to (\mathbf{A}_2, E^{\mathbf{A}_2}, \exists^{\mathbf{A}_2}),$$

define $ker(f) \subseteq \mathbf{A}_1$ *by*

$$ker(f) = \{p \in \mathbf{A}_1 \mid f(p) = \mathbf{0}^{\mathbf{A}_2}\}. \tag{2.3.1}$$

Lemma 2.3.12. *Let* $f : (\mathbf{A}_1, E^{\mathbf{A}_1}, \exists^{\mathbf{A}_1}) \to (\mathbf{A}_2, E^{\mathbf{A}_2}, \exists^{\mathbf{A}_2})$ *be an MBA-homomorphism. Then* $ker(f)$ *is an MBA-ideal in* $(\mathbf{A}_1, E^{\mathbf{A}_1}, \exists^{\mathbf{A}_1})$.

Proof. Let $p, q \in ker(f)$. Then $f(p) = f(q) = \mathbf{0}^{\mathbf{A}_2}$. So $f(p \vee q) = f(p) \vee f(q) = \mathbf{0}^{\mathbf{A}_2} \vee \mathbf{0}^{\mathbf{A}_2} = \mathbf{0}^{\mathbf{A}_2}$. Thus $p \vee q \in ker(f)$.

Let $p \in ker(f)$ and $q \in \mathbf{A}_1$. Then $f(p) = \mathbf{0}^{\mathbf{A}_2}$. So $f(p \wedge q) = f(p) \wedge f(q) = \mathbf{0}^{\mathbf{A}_2} \wedge f(q) = \mathbf{0}^{\mathbf{A}_2}$. Thus $p \wedge q \in ker(f)$.

Let $p \in ker(f)$. Then $f(p) = \mathbf{0}^{\mathbf{A}_2}$. Hence $f(\exists^{\mathbf{A}_1} p) = \exists^{\mathbf{A}_2} f(p) = \exists^{\mathbf{A}_2} \mathbf{0}^{\mathbf{A}_2} = \mathbf{0}^{\mathbf{A}_2}$. Thus $\exists^{\mathbf{A}_1} p \in ker(f)$. □

Suppose $(\mathbf{A}, E, \exists)$ is an MBA.

Definition 2.3.13. *For* $p \in \mathbf{A}$, *define a subset* $\Delta(p) \subseteq \mathbf{A}$ *by*

$$\Delta(p) = \{q \in \mathbf{A} \mid q \leq \exists p\}.$$

Lemma 2.3.14. $\Delta(p)$ *is an MBA-ideal.*

Proof. Let $q_0, q_1 \in \Delta(p)$. Then $q_0 \leq \exists p$ and $q_1 \leq \exists p$. So $q_0 \vee q_1 \leq \exists p \vee \exists p = \exists p$. Thus $q_0 \vee q_1 \in \Delta(p)$.

Let $q_0 \in \Delta(p)$ and $q_1 \in \mathbf{A}$. Then $q_0 \leq \exists p$. So $q_0 \wedge q_1 \leq q_0 \leq \exists p$. Thus $q_0 \wedge q_1 \in \Delta(p)$.

Let $q \in \Delta(p)$. Then $q \leq \exists p$. Hence $\exists q \leq \exists p$ (by Lemma 2.2.6 (2)). So $\exists q \in \Delta(p)$. □

Definition 2.3.15. *An MBA-ideal* Δ *is* ***virtual*** *iff* $p \wedge E = \mathbf{0}$ *for all* $p \in \Delta$.

Example 2.3.16. *Let* $p \in \mathbf{A}$ *and* $p \wedge E = \mathbf{0}$. *Then* $\Delta = \{q \in \mathbf{A} \mid q \leq p\}$ *is a virtual MBA-ideal. In particular,* $\{q \in \mathbf{A} \mid q \leq E'\}$ *is a virtual MBA-ideal and it is the biggest virtual MBA-ideal in* $(\mathbf{A}, E, \exists)$.

Definition 2.3.17. *An MBA* $(\mathbf{A}, E, \exists)$ *is* ***basic*** *iff every proper MBA-ideal in* $(\mathbf{A}, E, \exists)$ *is virtual.*

Lemma 2.3.18. *An MBA* $(\mathbf{A}, E, \exists)$ *is basic iff the quantifier* $\exists$ *is basic.*

Proof. Suppose $(\mathbf{A}, E, \exists)$ is basic. To be proved that $\exists$ is basic. Suppose $p \in \mathbf{A}$ and $p \wedge E \neq \mathbf{0}$. We will prove that $\exists p = \mathbf{1}$. Since $p \wedge E \leq \exists p$, we have $p \wedge E \in \Delta(p)$ $(= \{q \in \mathbf{A} \mid q \leq \exists p\})$. Let $\bar{p} = p \wedge E$. So $\bar{p} \in \Delta(p)$ and $\bar{p} \wedge E \neq \mathbf{0}$ (since $\bar{p} \wedge E = (p \wedge E) \wedge E = p \wedge E \neq \mathbf{0}$). Thus $\Delta(p)$ is not virtual. Hence $\Delta(p) = \mathbf{A}$ (since $\mathbf{A}$ is basic). Then $\mathbf{1} \in \Delta(p)$. Therefore $\mathbf{1} \leq \exists p$ (by definition of $\Delta(p)$). So $\exists p = \mathbf{1}$.

Suppose the quantifier $\exists$ is basic. To be proved that every non-virtual MBA-ideal in $(\mathbf{A}, E, \exists)$ is improper. Let Δ be an MBA-ideal which is not virtual. Then there exists $p \in \Delta$ such that $p \wedge E \neq \mathbf{0}$. Hence $\exists p = \mathbf{1}$ (since $\exists$ is basic). Then $\exists p \in \Delta$ (since Δ is an MBA-ideal and $p \in \Delta$) and $\mathbf{1} \in \Delta$. So $\Delta = \mathbf{A}$. Thus Δ is an improper MBA-ideal in $\mathbf{A}$. $\square$

Definition 2.3.19. *Let* X *be a set,* $X_E \subseteq X$ *and* E *be the characteristic function of* X_E*. A* ***model*** *is a* $\mathbf{2}$*-valued functional monadic bounded algebra with domain* (X, X_E) *and designated function* E*.*

Theorem 2.3.20 (cf. [7, Theorem 6])**.** *An MBA is basic if and only if it is (isomorphic to) a model (i.e. every basic MBA is isomorphic to a model and every model is basic).*

Proof. Let $\mathbf{M}$ be a model and $p \in \mathbf{M}$. Suppose $p \wedge E \neq \mathbf{0}$ (as functions from X to $\mathbf{2}$). Hence $p(x_0) \neq \mathbf{0}$ and $E(x_0) \neq \mathbf{0}$ for some $x_0 \in X$. Therefore $p(x_0) = \mathbf{1}$ and $E(x_0) = \mathbf{1}$. Hence $\bigvee\{E(y) \wedge p(y) \mid y \in X\} = \mathbf{1}$. Thus $\exists p = \mathbf{1}$ (as functions from X to $\mathbf{2}$). Suppose $p \wedge E = \mathbf{0}$ (as functions from X to $\mathbf{2}$). Hence $\bigvee\{E(y) \wedge p(y) \mid y \in X\} = \mathbf{0}$. Thus $\exists p = \mathbf{0}$ (as functions from X to $\mathbf{2}$). So we have proved that the functional quantifier $\exists$ of $\mathbf{M}$ is basic. Hence $\mathbf{M}$ is basic (by Lemma 2.3.18).

Now suppose $(\mathbf{A}, E, \exists)$ is a basic MBA. Note that $\mathbf{A}$ may be considered as a Boolean algebra. As in [7, p. 48], define:

- $W_{\mathbf{A}} = \{x \mid x$ is a (proper) Boolean ultrafilter of the Boolean algebra $\mathbf{A}\}$;
- a function $\varphi : \mathbf{A} \to \mathcal{P}(W_{\mathbf{A}})$ by $\varphi(p) = \{x \in W_{\mathbf{A}} \mid p \in x\}$ (for every $p \in \mathbf{A}$).

We are going to define desired sets X, X_E and a model in $\mathbf{2}^X$. Let

- $X = W_{\mathbf{A}}$ and
- $X_E = \{x \in X \mid E \in x\}$.

Define a mapping $H : \mathbf{A} \to \mathbf{2}^X$ by

$$H(p)(x) = \begin{cases} \mathbf{1}, & \text{if } x \in \varphi(p) \\ \mathbf{0}, & \text{if } x \notin \varphi(p) \end{cases}$$

for every $x \in X$ and $p \in \mathbf{A}$; and define $E^f \in \mathbf{2}^X$ by $E^f = H(E)$ (where the superscript f stands for the adjective "functional").

Then:

- H is one-to-one. Suppose $p_0, p_1 \in \mathbf{A}$ and $p_0 \neq p_1$. Hence there is $u \in W_{\mathbf{A}}$ such that $p_0 \in u$ and $p_1 \notin u$ (or, $p_0 \notin u$ and $p_1 \in u$). Then $u \in \varphi(p_0)$ and $u \notin \varphi(p_1)$ (or, $u \notin \varphi(p_0)$ and $u \in \varphi(p_1)$). So $H(p_0)(u) = \mathbf{1}$ and $H(p_1)(u) = \mathbf{0}$ (or, $H(p_0)(u) = \mathbf{0}$ and $H(p_1)(u) = \mathbf{1}$). Thus $H(p_0) \neq H(p_1)$.
- H preserves $\wedge, \vee, '$. It is proved by the properties of ultrafilters. We consider only $\wedge$ ($\vee$ and $'$ are proved similarly):

$$H(p \wedge q)(x) = \begin{cases} \mathbf{1}, & \text{if } x \in \varphi(p \wedge q) \\ \mathbf{0}, & \text{if } x \notin \varphi(p \wedge q) \end{cases} = \begin{cases} \mathbf{1}, & \text{if } p \wedge q \in x \\ \mathbf{0}, & \text{if } p \wedge q \notin x \end{cases}$$

$$= \begin{cases} \mathbf{1}, & \text{if } p \in x \text{ and } q \in x \\ \mathbf{0}, & \text{if } p \notin x \text{ or } q \notin x \end{cases} = \begin{cases} \mathbf{1}, & \text{if } p \in x \text{ and } q \in x \\ \mathbf{0}, & \text{if } p \notin x \text{ and } q \in x \\ \mathbf{0}, & \text{if } p \in x \text{ and } q \notin x \\ \mathbf{0}, & \text{if } p \notin x \text{ and } q \notin x \end{cases}$$

$$= H(p)(x) \wedge H(q)(x) = (H(p) \wedge H(q))(x).$$

So $H(p \wedge q) = H(p) \wedge H(q)$.

Now consider

$$E^f(x) = H(E)(x) = \begin{cases} \mathbf{1}, & \text{if } x \in \varphi(E) \\ \mathbf{0}, & \text{if } x \notin \varphi(E) \end{cases} = \begin{cases} \mathbf{1}, & \text{if } x \in X_E \\ \mathbf{0}, & \text{if } x \notin X_E \end{cases}.$$

So E^f is the characteristic function of the subset $X_E \subseteq X$.

It remains to prove that

$$H(\exists p) = \exists H(p) \tag{2.3.2}$$

for every $p \in \mathbf{A}$. Note that the quantifier on the right side is functional, i.e. $\exists H(p)(x_0) = \bigvee\{H(p)(x) \mid x \in X_E\}$ for $x_0 \in X$.

Since $(\mathbf{A}, E, \exists)$ is basic, the quantifier of $(\mathbf{A}, E, \exists)$ is basic. Hence, for every $p \in \mathbf{A}$,

$$\exists p = \begin{cases} \mathbf{1}, & \text{if } p \wedge E \neq \mathbf{0} \text{ (or } p \nleqslant E') \\ \mathbf{0}, & \text{if } p \wedge E = \mathbf{0} \text{ (or } p \leqslant E'). \end{cases} \tag{2.3.3}$$

To prove (2.3.2) we will consider two cases.

Case 1 $p \wedge E \neq \mathbf{0}$. Then $\exists p = \mathbf{1}$. Therefore, for every $x \in X$, $H(\exists p)(x) = H(\mathbf{1})(x) = \begin{cases} \mathbf{1}, & \text{if } x \in \varphi(\mathbf{1}) \\ \mathbf{0}, & \text{if } x \notin \varphi(\mathbf{1}) \end{cases} = \begin{cases} \mathbf{1}, & \text{if } \mathbf{1} \in x \\ \mathbf{0}, & \text{if } \mathbf{1} \notin x \end{cases}$ [by definition of φ] $=$ $\mathbf{1}$ [since the unit element belongs to every filter]. Now consider the right side of (2.3.2). Since $H(p)(x) = \mathbf{1}$ iff $x \in \varphi(p)$, we can write

$$\exists H(p)(x) = \begin{cases} \mathbf{1}, & \text{if there is } y \in X_E \text{ such that } y \in \varphi(p) \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{2.3.4}$$

Since $p \wedge E \neq \mathbf{0}$, the (principal Boolean) filter $\Delta = \{q \in \mathbf{A} \mid q \geqslant p \wedge E\}$ is proper. Then there is an (proper) ultrafilter Δ_0 (in the Boolean algebra $\mathbf{A}$) with $\Delta \subseteq \Delta_0$ (so $\Delta_0 \in X$). Hence $p \wedge E \in \Delta_0$. Then both $p \in \Delta_0$ and $E \in \Delta_0$ (since Δ_0 is an ultrafilter). Therefore $\Delta_0 \in X_E$ (by definition of X_E) and $\Delta_0 \in \varphi(p)$ (by definition of φ). So $\exists H(p)(x) = \mathbf{1}$

for every $x \in X$ (by (2.3.4)). (There is a shorter proof. From $p \wedge E \neq \mathbf{0}$ follows $H(p) \wedge E^f \neq \mathbf{0}$ (since H is one-to-one and preserves $\wedge$); hence $\exists H(p) = \mathbf{1}$ (as functions from X to $\mathbf{2}$) by the first part of this proof). Thus (in this case) we have proved that $H(\exists p)(x) = \mathbf{1}$ and $\exists H(p)(x) = \mathbf{1}$ for every $x \in X$. Hence $H(\exists p) = \exists H(p)$.

Case 2 $p \wedge E = \mathbf{0}$. Then $\exists p = \mathbf{0}$. Therefore, for every $x \in X$, $H(\exists p)(x) = H(\mathbf{0})(x) = \begin{cases} \mathbf{1}, & \text{if } x \in \varphi(\mathbf{0}) \\ \mathbf{0}, & \text{if } x \notin \varphi(\mathbf{0}) \end{cases} = \begin{cases} \mathbf{1}, & \text{if } \mathbf{0} \in x \\ \mathbf{0}, & \text{if } \mathbf{0} \notin x \end{cases}$ [by definition of φ] $= \mathbf{0}$ (since there is no (proper Boolean) ultrafilter containing the zero element). Now consider the right side of (2.3.2). We claim that there is no $y \in X_E$ such that $y \in \varphi(p)$ (cf. (2.3.4)). Assume there is $y_0 \in X_E$ with $y_0 \in \varphi(p)$. Hence $p \in y_0$ (by definition of φ) and $E \in y_0$ (by definition of X_E). Then $p \wedge E \in y_0$ (since y_0 is an ultrafilter). Since $p \wedge E = \mathbf{0}$ (in this case), we obtain $\mathbf{0} \in y_0$. Thus y_0 is not proper. So $\exists H(p)(x) = \mathbf{0}$ for every $x \in X$ (by (2.3.4)). (As in Case 1, there is a shorter proof. From $p \wedge E = \mathbf{0}$ follows $H(p) \wedge E^f = \mathbf{0}$ (since H is one-to-one and preserves $\wedge$); hence $\exists H(p) = \mathbf{0}$ (as functions from X to $\mathbf{2}$)). Thus (in this case) we have proved that $H(\exists p)(x) = \mathbf{0}$ and $\exists H(p)(x) = \mathbf{0}$ for every $x \in X$. Hence $H(\exists p) = \exists H(p)$.

It follows from Case 1 and Case 2 that $H(\exists p) = \exists H(p)$.

So $H : \mathbf{A} \to \mathbf{2}^X$ is an one-to-one mapping which preserves $\wedge, \vee, ', E$, and $\exists$. Thus $H(\mathbf{A}) \subseteq \mathbf{2}^X$ is our desired model. $\square$

2.4 Representation of an MBA as a subdirect product of basic MBA's

In this section we prove the second representation theorem. The theorem is based, among other things, on E.J. Lemmon's representation theorem of modal algebras [8, p. 206] and on the notion of the reflexive-transitive closure of a relation from modal logic [5, p. 9-10]. The theorem will help us to analyze MBA-varieties in Chapter 3. In the end of this section we discuss an algebraic version of the completeness theorem for free monadic logic as well as the representation theorem from the point of view of subdirect irreducibility.

Definition 2.4.1. *Let* $((\mathbf{A}_i, E_i, \exists_i))_{i \in I}$ *be an indexed family of MBA's. The* ***direct product*** $(\mathbf{A}, E, \exists)$ *of the family is an MBA with the universe* $\prod_{i \in I} \mathbf{A}_i$ *whose operations are defined coordinate-wise. The empty product* $\prod \emptyset$ *is the trivial MBA, i.e. one element MBA, with the universe* $\{\emptyset\}$*. For every* $j \in I$*, define the projection map*

$$\pi_j : \prod_{i \in I} \mathbf{A}_i \to \mathbf{A}_j$$

by

$$\pi_j(a) = a(j)$$

$\big($*for every* $a \in \prod_{i \in I} \mathbf{A}_i\big)$.

Definition 2.4.2. *An MBA* $(\mathbf{A}, E, \exists)$ *is a* ***subdirect product*** *of an indexed family* $((\mathbf{A}_i, E_i, \exists_i))_{i \in I}$ *of MBA's iff*

- $\mathbf{A}$ *is an MBA-subalgebra of the direct product* $\prod_{i \in I} \mathbf{A}_i$*,*
- $\pi_i(\mathbf{A}) = \mathbf{A}_i$ *for each* $i \in I$*.*

An embedding $f : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$ *is* ***subdirect*** *if* $f(\mathbf{A})$ *is a subdirect product of the* $(\mathbf{A}_i)_{i \in I}$*.*

Definition 2.4.3. *An MBA* $\mathbf{A}$ *is* ***subdirectly irreducible*** *if for every subdirect embedding*

$$f : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$$

there is an $i \in I$ *such that*

$$\pi_i \circ f : \mathbf{A} \to \mathbf{A}_i$$

is an MBA-isomorphism.

Let us reformulate Theorem 8.4 from [3]:

Theorem 2.4.4. *An MBA* $\mathbf{A}$ *is subdirectly irreducible iff* $\mathbf{A}$ *is trivial or there is a smallest non-identity congruence on* $\mathbf{A}$.

Definition 2.4.5. *If we are given maps* $f_i : X \to X_i$, $i \in I$, *then the* ***natural map***

$$f : X \to \prod_{i \in I} X_i$$

is defined by

$$(f(x))(i) = f_i(x),$$

for every $x \in X$ *and* $i \in I$.

The next lemma is known from universal algebra ([3, Theorem 7.12(a)]).

Lemma 2.4.6. *If* $f_i : (\mathbf{A}, E, \exists) \to (\mathbf{A}_i, E_i, \exists_i)$, $i \in I$, *is an indexed family of MBA-homomorphisms, then the natural map* f *is an MBA-homomorphism from* $\mathbf{A}$ *to* $\mathbf{A}^* = \prod_{i \in I} \mathbf{A}_i$.

Proof. For all $p, q \in \mathbf{A}$ and $i \in I$,$f(p \wedge q)(i) = f_i(p \wedge q) = f_i(p) \wedge f_i(q) = (f(p))(i) \wedge (f(q)(i)) = (f(p) \wedge f(q))(i)$. Hence $f(p \wedge q) = f(p) \wedge f(q)$.

For all $p \in \mathbf{A}$ and $i \in I$, $f(p')(i) = f_i(p') = (f_i(p))' = (f(p)(i))' = (f(p))'(i)$. Hence $f(p') = (f(p))'$.

Since $f(E)(i) = f_i(E) = E_i$ for every $i \in I$, we have $f(E) = E^{\mathbf{A}^*}$.

For all $p \in \mathbf{A}$ and $i \in I$, $f(\exists p)(i) = f_i(\exists p) = \exists_i(f_i(p)) = \exists_i(f(p)(i)) = (\exists^{\mathbf{A}^*} f(p))(i)$. Hence $f(\exists p) = \exists^{\mathbf{A}^*} f(p)$.

Thus f is an MBA-homomorphism. □

The notion of reflexive-transitive closure from modal logic (see [5, p. 9-10]) will be useful in the second representation theorem.

Definition 2.4.7. *Let* (W, R, E) *be a marked directed graph. Define on* W *the relations* $R^n \subseteq W \times W$, *for* $n \geq 0$, *and* R^*, *as follows:*

- uR^0v *iff* $u = v$,
- $uR^{n+1}v$ *iff there exists* w *such that both* uR^nw *and* wRv,
- $R^* = \bigcup_{n\geq 0} R^n$.

It is easy to see that R^* *is the smallest reflexive-transitive relation including* R*, and therefore* R^* *is called the* ***reflexive-transitive closure*** *of* R*.*

Lemma 2.4.8. *Let* (W, R, E) *be a marked directed graph such that* R *is transitive and* $x \in W$*. Put* $W^x = \{u \in W \mid xR^*u\}$*. Then* $W^x = \{x\} \cup \{u \in W \mid xRu\}$*.*

Proof. For $\subseteq$, suppose $v \in W^x$. Then xR^*v. If xR^0v, then $x = v$. Hence $v \in \{x\} \cup \{u \in W \mid xRu\}$. If xR^nv for some $1 \leq n < \omega$, then xRv (since R is transitive). Hence $v \in \{x\} \cup \{u \in W \mid xRu\}$. For $\supseteq$, suppose $v \in \{x\} \cup \{u \in W \mid xRu\}$. If $v = x$, then xR^0v. Hence xR^*v. Therefore $v \in W^x$. If xRv, then xR^*v. Hence $v \in W^x$. □

We are now ready to prove the main result of the section.

Theorem 2.4.9. *Every MBA is isomorphic to a subdirect product of basic MBA's.*

Proof. Suppose $(\mathbf{A}, E, \exists)$ is an MBA. As in [8, p. 206], define:

1. $W_{\mathbf{A}} = \{x \mid x$ is an (proper Boolean) ultrafilter of the Boolean algebra $\mathbf{A}\}$;
2. A relation $R_{\mathbf{A}}$ on $W_{\mathbf{A}}$ by

$$xR_{\mathbf{A}}y \text{ iff } \{\exists p \mid p \in y\} \subseteq x,$$

   for every $x, y \in W_{\mathbf{A}}$;

3. A function $\varphi : \mathbf{A} \to \mathcal{P}(W_{\mathbf{A}})$ by

$$\varphi(p) = \{x \in W_{\mathbf{A}} \mid p \in x\},$$

   for every $p \in \mathbf{A}$;

4. $\mathbf{P_A} = \{\varphi(p) \mid p \in \mathbf{A}\}$ (so $\mathbf{P_A} \subseteq \mathcal{P}(W_\mathbf{A})$);

5. An operator $\langle R_\mathbf{A}\rangle : \mathcal{P}(W_\mathbf{A}) \to \mathcal{P}(W_\mathbf{A})$ by

$$\langle R_\mathbf{A}\rangle C = \{x \in W_\mathbf{A} \mid \text{there is } y \in W_\mathbf{A} \text{ such that } y \in C \text{ and } xR_\mathbf{A}y\},$$

   for every $C \in \mathcal{P}(W_\mathbf{A})$;

6. $E_\mathbf{A} \in \mathbf{P_A}$ by $E_\mathbf{A} = \varphi(E)$.

It follows from [8, p. 206] that φ is an MBA-isomorphism between $(\mathbf{A}, E, \exists)$ and $(\mathbf{P_A}, E_\mathbf{A}, \langle R_\mathbf{A}\rangle)$.

For $x \in W_\mathbf{A}$, define $W^x_\mathbf{A} = \{y \in W_\mathbf{A} \mid xR^*_\mathbf{A}y\}$, where $R^*_\mathbf{A}$ is the reflexive-transitive closure of $R_\mathbf{A}$. Also, for every $x \in W_\mathbf{A}$, define $\mathbf{P}^x_\mathbf{A} = \{W^x_\mathbf{A} \cap X \mid X \in \mathbf{P_A}\}$ (so $\mathbf{P}^x_\mathbf{A} \subseteq \mathcal{P}(W^x_\mathbf{A})$), $E^x_\mathbf{A} = W^x_\mathbf{A} \cap E_\mathbf{A}$, and $R^x_\mathbf{A} = R_\mathbf{A} \cap (W^x_\mathbf{A} \times W^x_\mathbf{A})$.

Note that $E^x_\mathbf{A} \in \mathbf{P}^x_\mathbf{A}$ for $x \in W_\mathbf{A}$. For $x \in W_\mathbf{A}$, define a mapping $\theta_x : \mathbf{P_A} \to \mathbf{P}^x_\mathbf{A}$ by

$$\theta_x(X) = W^x_\mathbf{A} \cap X,$$

for every $X \in \mathbf{P_A}$.

Obviously, θ_x is onto and $\theta_x(E_\mathbf{A}) = E^x_\mathbf{A}$.

Moreover, θ_x preserves Boolean operations, because $\theta_x(X \cap Y) = W^x_\mathbf{A} \cap (X \cap Y) = (W^x_\mathbf{A} \cap X) \cap (W^x_\mathbf{A} \cap Y) = \theta_x(X) \cap \theta_x(Y)$, $\theta_x(X \cup Y) = W^x_\mathbf{A} \cap (X \cup Y) = (W^x_\mathbf{A} \cap X) \cup (W^x_\mathbf{A} \cap Y) = \theta_x(X) \cup \theta_x(Y)$ and $\theta_x(W_\mathbf{A} - X) = W^x_\mathbf{A} \cap (W_\mathbf{A} - X) = W^x_\mathbf{A} - X = W^x_\mathbf{A} - (W^x_\mathbf{A} \cap X) = W^x_\mathbf{A} - \theta_x(X)$.

We are going to prove that $\theta_x(\langle R_\mathbf{A}\rangle X) = \langle R^x_\mathbf{A}\rangle \theta_x(X)$, i.e.

$$W^x_\mathbf{A} \cap (\langle R_\mathbf{A}\rangle X) = \langle R^x_\mathbf{A}\rangle (W^x_\mathbf{A} \cap X).$$

For $\subseteq$ part, suppose $u \in W^x_\mathbf{A} \cap (\langle R_\mathbf{A}\rangle X)$. Hence $u \in W^x_\mathbf{A}$ and $uR_\mathbf{A}v$ for some $v \in X$. Since $u \in W^x_\mathbf{A}$, we have $xR^*_\mathbf{A}u$. Then $xR^*_\mathbf{A}v$ (since $R^*_\mathbf{A}$ is the reflexive-transitive closure of $R_\mathbf{A}$ and $uR_\mathbf{A}v$). So $v \in W^x_\mathbf{A}$ (and $v \in W^x_\mathbf{A} \cap X$). Thus $\langle u, v\rangle \in R_\mathbf{A} \cap (W^x_\mathbf{A} \times W^x_\mathbf{A})$. Hence $uR^x_\mathbf{A}v$. Then $u \in \langle R^x_\mathbf{A}\rangle (W^x_\mathbf{A} \cap X)$ (since $v \in W^x_\mathbf{A} \cap X$). For $\supseteq$ part, suppose $u \in \langle R^x_\mathbf{A}\rangle (W^x_\mathbf{A} \cap X)$. Hence $uR^x_\mathbf{A}v$ for some $v \in W^x_\mathbf{A} \cap X$. Since $uR^x_\mathbf{A}v$, we have $\langle u, v\rangle \in R_\mathbf{A} \cap (W^x_\mathbf{A} \times W^x_\mathbf{A})$. Then

$u, v \in W^x_{\mathbf{A}}$ and $uR_{\mathbf{A}}v$. It follows from $uR_{\mathbf{A}}v$ and $v \in X$ that $u \in \langle R_{\mathbf{A}}\rangle X$. So $u \in W^x_{\mathbf{A}} \cap (\langle R_{\mathbf{A}}\rangle X)$ (since $u \in W^x_{\mathbf{A}}$).

Thus, for every $x \in W_{\mathbf{A}}$, $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$ is an MBA and

$$\theta_x : (\mathbf{P}_{\mathbf{A}}, E_{\mathbf{A}}, \langle R_{\mathbf{A}}\rangle) \to (\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$$

is a surjective MBA-homomorphism.

Now define

$$\Theta : \mathbf{P}_{\mathbf{A}} \to \prod_{x \in W_{\mathbf{A}}} \mathbf{P}^x_{\mathbf{A}}$$

by

$$\Theta(X)(x) = \theta_x(X),$$

for every $X \in \mathbf{P}_{\mathbf{A}}$ and $x \in W_{\mathbf{A}}$ (so Θ is the natural map).

Then we can obtain the following:

- Θ is one-to-one. Suppose $X_0, X_1 \in \mathbf{P}_{\mathbf{A}}$ and $X_0 \neq X_1$. Hence there is $x \in W_{\mathbf{A}}$ such that $x \in X_0$ and $x \notin X_1$ (or, $x \notin X_0$ and $x \in X_1$). Then $x \in W^x_{\mathbf{A}} \cap X_0$ and $x \notin W^x_{\mathbf{A}} \cap X_1$ (or, $x \notin W^x_{\mathbf{A}} \cap X_0$ and $x \in W^x_{\mathbf{A}} \cap X_1$) (since $x \in W^x_{\mathbf{A}}$). Therefore $\theta_x(X_0) \neq \theta_x(X_1)$. So $\Theta(X_0)(x) \neq \Theta(X_1)(x)$. Thus $\Theta(X_0) \neq \Theta(X_1)$.

- Θ is an MBA-homomorphism (by Lemma 2.4.6).

- For every $x \in W_{\mathbf{A}}$, the projection map $\pi_x : \Theta(\mathbf{P}_{\mathbf{A}}) \to \mathbf{P}^x_{\mathbf{A}}$ is onto. Suppose $Y \in \mathbf{P}^x_{\mathbf{A}}$. Hence $Y = \theta_x(X)$ for some $X \in \mathbf{P}^x_{\mathbf{A}}$ (since θ_x is onto). Then $Y = W^x_{\mathbf{A}} \cap X$. Consider $\pi_x(\Theta(X)) = \Theta(X)(x)$[by definition of projection] = $\theta_x(X) = W^x_{\mathbf{A}} \cap X = Y$. So π_x is onto.

Therefore the image of the MBA $(\mathbf{P}_{\mathbf{A}}, E_{\mathbf{A}}, \langle R_{\mathbf{A}}\rangle)$ under Θ is a subdirect product of the family of MBA's $\{(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle) \mid x \in W_{\mathbf{A}}\}$. So the MBA $(\mathbf{P}_{\mathbf{A}}, E_{\mathbf{A}}, \langle R_{\mathbf{A}}\rangle)$ itself is isomorphic to a subdirect product of the MBA's $\{(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle) \mid x \in W_{\mathbf{A}}\}$. Hence $(\mathbf{A}, E, \exists)$ is isomorphic to a subdirect product of the MBA's $\{(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle) \mid x \in W_{\mathbf{A}}\}$.

Next we are going to prove that $(W_{\mathbf{A}}, R_{\mathbf{A}}, E_{\mathbf{A}})$ is a bounded graph.

- $R_{\mathbf{A}}$ is transitive. Suppose $x, y, z \in W_{\mathbf{A}}$ are such that $xR_{\mathbf{A}}y$ and $yR_{\mathbf{A}}z$. Then $\{\exists p \mid p \in y\} \subseteq x$ and $\{\exists p \mid p \in z\} \subseteq y$ (by definition of $R_{\mathbf{A}}$). Assume $p \in z$. Then $\exists p \in y$, and so $\exists\exists p \in x$. Therefore $\exists p \in x$ (since $\exists\exists p = \exists p$). So $\{\exists p \mid p \in z\} \subseteq x$, i.e. $xR_{\mathbf{A}}z$.

- $R_{\mathbf{A}}$ is Euclidean. Suppose $x, y, z \in W_{\mathbf{A}}$ are such that $xR_{\mathbf{A}}y$ and $xR_{\mathbf{A}}z$. Then $\{\exists p \mid p \in y\} \subseteq x$ and $\{\exists p \mid p \in z\} \subseteq x$. Suppose $p \in z$. We claim that $\exists p \in y$. Assume $\exists p \notin y$. Then $(\exists p)' \in y$ (since y is an ultrafilter). Therefore $\exists(\exists p)' \in x$. Hence $(\exists p)' \in x$ (by Lemma 2.2.6(10)). On the other hand, from $p \in z$ follows that $\exists p \in x$. Thus both $(\exists p)' \in x$ and $\exists p \in x$, and hence x is not proper. So $\{\exists p \mid p \in z\} \subseteq y$, i.e. $yR_{\mathbf{A}}z$.

- $\forall x, y \in W_{\mathbf{A}}(xR_{\mathbf{A}}y \to y \in E_{\mathbf{A}})$. Suppose $x, y \in W_{\mathbf{A}}$ and $xR_{\mathbf{A}}y$. Then $\{\exists p \mid p \in y\} \subseteq x$. We claim that $E \in y$ (and so $y \in E_{\mathbf{A}}$). Assume $E \notin y$. Then $E' \in y$ (since y is an ultrafilter). Hence $\exists E' \in x$ (by assumption). Therefore $\mathbf{0} \in x$ (by Lemma 2.2.6(5)). Thus x is not proper. So $y \in E_{\mathbf{A}}$.

- $\forall x \in W_{\mathbf{A}}(x \in E_{\mathbf{A}} \to xR_{\mathbf{A}}x)$. Suppose $x \in W_{\mathbf{A}}$ and $x \in E_{\mathbf{A}}$. Then $E \in x$ (by definition of $E_{\mathbf{A}}$). Suppose $p \in x$. Then $p \wedge E \in x$ (since $E \in x$ and x is an ultrafilter). Therefore $\exists p \in x$ (since $p \wedge E \leq \exists p$). So $\{\exists p \mid p \in x\} \subseteq x$, i.e. $xR_{\mathbf{A}}x$.

We are going to examine structures $(W^x_{\mathbf{A}}, R^x_{\mathbf{A}}, E^x_{\mathbf{A}})$, $x \in W_{\mathbf{A}}$. Recall that the goal is to prove that the MBA's $\mathbf{P}^x_{\mathbf{A}}$, $x \in W_{\mathbf{A}}$, are basic. Note that since $R_{\mathbf{A}}$ is transitive we can write, by Lemma 2.4.8,

$$W^x_{\mathbf{A}} = \{x\} \cup \{y \in W_{\mathbf{A}} \mid xR_{\mathbf{A}}y\}.$$

There are two cases.

Case 1 $xR^x_{\mathbf{A}}x$. To be proved that $E^x_{\mathbf{A}} = W^x_{\mathbf{A}}$ and $R^x_{\mathbf{A}} = W^x_{\mathbf{A}} \times W^x_{\mathbf{A}}$. Since $xR^x_{\mathbf{A}}x$, we have $W^x_{\mathbf{A}} = \{y \in W_{\mathbf{A}} \mid xR_{\mathbf{A}}y\}$. By definition of $E^x_{\mathbf{A}}$, $E^x_{\mathbf{A}} \subseteq W^x_{\mathbf{A}}$. For the other direction, suppose $u \in W^x_{\mathbf{A}}$. Then $xR_{\mathbf{A}}u$. Therefore $u \in E_{\mathbf{A}}$ (by the third property of $R_{\mathbf{A}}$). Thus $u \in W^x_{\mathbf{A}} \cap E_{\mathbf{A}}$, i.e. $u \in$

$E^x_{\mathbf{A}}$. So $E^x_{\mathbf{A}} = W^x_{\mathbf{A}}$. By definition of $R^x_{\mathbf{A}}$, $R^x_{\mathbf{A}} \subseteq W^x_{\mathbf{A}} \times W^x_{\mathbf{A}}$. For the other direction, suppose $\langle u, v\rangle \in W^x_{\mathbf{A}} \times W^x_{\mathbf{A}}$. Then $\langle x, u\rangle \in R_{\mathbf{A}}$ and $\langle x, v\rangle \in R_{\mathbf{A}}$. Therefore $\langle u, v\rangle \in R_{\mathbf{A}}$ (since $R_{\mathbf{A}}$ is Euclidean). Thus $\langle u, v\rangle \in R^x_{\mathbf{A}}$. So $R^x_{\mathbf{A}} = W^x_{\mathbf{A}} \times W^x_{\mathbf{A}}$.

Case 2 $\langle x, x\rangle \notin R^x_{\mathbf{A}}$. To be proved that $E^x_{\mathbf{A}} = W^x_{\mathbf{A}} - \{x\}$ and $R^x_{\mathbf{A}}$ is universal on $E^x_{\mathbf{A}}$, i.e. $R^x_{\mathbf{A}} \cap (E^x_{\mathbf{A}} \times E^x_{\mathbf{A}}) = E^x_{\mathbf{A}} \times E^x_{\mathbf{A}}$ (and so $R^x_{\mathbf{A}} = \{\langle x, y\rangle \mid y \in E^x_{\mathbf{A}}\} \cup (E^x_{\mathbf{A}} \times E^x_{\mathbf{A}})$). From $\langle x, x\rangle \notin R^x_{\mathbf{A}}$ and $\langle x, x\rangle \in W^x_{\mathbf{A}} \times W^x_{\mathbf{A}}$ follows that $\langle x, x\rangle \notin R_{\mathbf{A}}$. Suppose $u \in W^x_{\mathbf{A}} - \{x\}$. Then $u \in W^x_{\mathbf{A}}$. Therefore $xR_{\mathbf{A}}u$. Hence $u \in E_{\mathbf{A}}$ (by the third property of $R_{\mathbf{A}}$). Thus $u \in W^x_{\mathbf{A}} \cap E_{\mathbf{A}}$. Then $u \in E^x_{\mathbf{A}}$ (by definition of $E^x_{\mathbf{A}}$). So $W^x_{\mathbf{A}} - \{x\} \subseteq E^x_{\mathbf{A}}$. Since $E^x_{\mathbf{A}} = W^x_{\mathbf{A}} \cap E_{\mathbf{A}}$, we have $E^x_{\mathbf{A}} \subseteq W^x_{\mathbf{A}}$. Hence it remains to prove that $x \notin E^x_{\mathbf{A}}$. From $\langle x, x\rangle \notin R_{\mathbf{A}}$ follows $x \notin E_{\mathbf{A}}$ (by the fourth property of $R_{\mathbf{A}}$). Therefore $x \notin W^x_{\mathbf{A}} \cap E_{\mathbf{A}}$. Then $x \notin E^x_{\mathbf{A}}$. Thus $E^x_{\mathbf{A}} \subseteq W^x_{\mathbf{A}} - \{x\}$. So $E^x_{\mathbf{A}} = W^x_{\mathbf{A}} - \{x\}$.
To be proved that $R^x_{\mathbf{A}} \cap (E^x_{\mathbf{A}} \times E^x_{\mathbf{A}}) = E^x_{\mathbf{A}} \times E^x_{\mathbf{A}}$. The $\subseteq$ part is obvious. For the $\supseteq$ part, suppose $\langle u, v\rangle \in E^x_{\mathbf{A}} \times E^x_{\mathbf{A}}$. Hence $u, v \in W^x_{\mathbf{A}}$ and $u \neq x$, $v \neq x$ (since $E^x_{\mathbf{A}} = W^x_{\mathbf{A}} - \{x\}$). Then $xR_{\mathbf{A}}u$ and $xR_{\mathbf{A}}v$. Therefore $uR_{\mathbf{A}}v$ (since $R_{\mathbf{A}}$ is Euclidean). Thus $\langle u, v\rangle \in (W^x_{\mathbf{A}} \times W^x_{\mathbf{A}}) \cap R_{\mathbf{A}}$, i.e. $\langle u, v\rangle \in R^x_{\mathbf{A}}$. So $\langle u, v\rangle \in R^x_{\mathbf{A}} \cap (E^x_{\mathbf{A}} \times E^x_{\mathbf{A}})$.

Finally, we are going to prove that the MBA's $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$, $x \in W_{\mathbf{A}}$, are basic. Recall that

$$\langle R^x_{\mathbf{A}}\rangle X = \{u \in W^x_{\mathbf{A}} \mid \text{ there is } v \in W^x_{\mathbf{A}} \text{ such that } uR^x_{\mathbf{A}}v \text{ and } v \in X\},$$

for every $X \in \mathbf{P}^x_{\mathbf{A}}$. Fix $x \in W_{\mathbf{A}}$. Let $X \in \mathbf{P}^x_{\mathbf{A}}$ and $X \cap E^x_{\mathbf{A}} \neq \emptyset$. To be proved that $\langle R^x_{\mathbf{A}}\rangle X = W^x_{\mathbf{A}}$. Since $\langle R^x_{\mathbf{A}}\rangle$ is an operation of the MBA $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$ and $W^x_{\mathbf{A}}$ is the unit element in $\mathbf{P}^x_{\mathbf{A}}$, we have $\langle R^x_{\mathbf{A}}\rangle X \subseteq W^x_{\mathbf{A}}$. For the other part, suppose $v \in W^x_{\mathbf{A}}$. Since $X \cap E^x_{\mathbf{A}} \neq \emptyset$, there is at least one element $u \in X \cap E^x_{\mathbf{A}}$. Hence $u \in X$ and $u \in E^x_{\mathbf{A}}$. It follows from $v \in W^x_{\mathbf{A}}$ and $u \in E^x_{\mathbf{A}}$ that $\langle v, u\rangle \in R^x_{\mathbf{A}}$ (in the both cases above). Hence $v \in \langle R^x_{\mathbf{A}}\rangle X$. So $\langle R^x_{\mathbf{A}}\rangle X = W^x_{\mathbf{A}}$. Thus $\langle R^x_{\mathbf{A}}\rangle$ is basic. Consequently, $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$ is basic (by Lemma 2.3.18).

Thus every MBA is isomorphic to a subdirect product of basic MBA's. □

Corollary 2.4.10. *Every MBA is isomorphic to a subdirect product of models.*

Proof. Follows from Thereom 2.3.20 and Theorem 2.4.9. □

Corollary 2.4.10 may be considered as an algebraic version of the completeness theorem for free monadic logic. In logic the completeness theorem says that

$$\Gamma \models \varphi \text{ implies } \Gamma \vdash \varphi. \tag{2.4.1}$$

where, as usual, $\Gamma \models \varphi$ means that Γ logically implies φ and $\Gamma \vdash \varphi$ means that φ is deriveable from Γ. To illustrate the corollary we should come back to our motivating examples $\mathrm{M}_0^{\mathfrak{A}}$ and M_1^C from Chapter 1.

As P. Halmos [7, p. 48], we work with the so called refutability rather than provability. Let Γ be a fixed set of sentences in free monadic language. We claim that $\Delta_{\mathbf{M}_1^C,\Gamma} = \{f(\varphi) \mid \Gamma \vdash \neg\varphi\}$ is an MBA-ideal in M_1^C (cf. [7, p. 48]). Suppose $f(\varphi_0), f(\varphi_1) \in \mathrm{M}_1^C$ are such that $\Gamma \vdash \neg\varphi_0$ and $\Gamma \vdash \neg\varphi_1$. Hence $\Gamma \vdash \neg\varphi_0 \wedge \neg\varphi_1$ and so $\Gamma \vdash \neg(\varphi_0 \vee \varphi_1)$. Then $f(\varphi_0 \vee \varphi_1) \in \{f(\varphi) \mid \Gamma \vdash \neg\varphi\}$. Thus $f(\varphi_0) \vee f(\varphi_1) \in \Delta_{\mathbf{M}_1^C,\Gamma}$ (since $f(\varphi_0 \vee \varphi_1) = f(\varphi_0) \vee f(\varphi_1)$). Now suppose $f(\varphi_0) \in \mathrm{M}_1^C$ is such that $\Gamma \vdash \neg\varphi_0$. Hence $\Gamma \vdash \neg\varphi_0 \vee \neg\psi$ (for any ψ) and so $\Gamma \vdash \neg(\varphi_0 \wedge \psi)$. Then $f(\varphi_0 \wedge \psi) \in \{f(\varphi) \mid \Gamma \vdash \neg\varphi\}$. Thus $f(\varphi_0) \wedge f(\psi) \in \Delta_{\mathbf{M}_1^C,\Gamma}$. Finally, suppose $f(\varphi_0) \in \mathrm{M}_1^C$ is such that $\Gamma \vdash \neg\varphi_0$. Hence $\Gamma \vdash \forall x \neg\varphi_0$ and so $\Gamma \vdash \neg\exists x \neg\neg\varphi_0$. Therefore $\Gamma \vdash \neg\exists x\varphi_0$. Then $f(\exists x\varphi_0) \in \{f(\varphi) \mid \Gamma \vdash \neg\varphi\}$. Thus $\exists f(\varphi_0) \in \Delta_{\mathbf{M}_1^C,\Gamma}$ (here $\exists$ is the functional quantifier of M_1^C).

Therefore the next definition is justified.

Definition 2.4.11 (cf. [7, p. 48]). *An **MBA-logic** is a pair $(\mathbf{A}, \mathbf{I})$, where $\mathbf{A}$ is an MBA and $\mathbf{I}$ is an MBA-ideal in $\mathbf{A}$. The elements $p \in \mathbf{I}$ are the **refutable** elements of the logic.*

Thus, if we have an MBA-logic $(\mathbf{A}, \mathbf{I})$, then we can form the quotient MBA $\mathbf{A}/\mathbf{I} = \{[p] \mid p \in \mathbf{A}\}$ where $[p] = \{q \in \mathbf{A} \mid p+q \in \mathbf{I}\}$ (see Lemma 2.3.8). Therefore, for every $p_0 \in \mathbf{A}$,

- if $p_0 \in \mathbf{I}$, then $[p_0] = \mathbf{0}$ ($\in \mathbf{A}/\mathbf{I}$);
- if $p_0 \notin \mathbf{I}$, then $[p_0] \neq \mathbf{0}$ ($\in \mathbf{A}/\mathbf{I}$).

Now we are going to analyze the left side of (2.4.1) (it assumed that the soundness theorem is known). $\Gamma \models \varphi$ says that, for every structure $\mathfrak{A}$, $\mathfrak{A} \models \Gamma$ implies $\mathfrak{A} \models \varphi$. Recall that every structure $\mathfrak{A}$ provides us with the model $\mathbf{M}_0^{\mathfrak{A}}$. So we have:

$\mathfrak{A} \models \Gamma$ iff $\mathfrak{A} \models \psi$, for all ψ such that $\Gamma \vdash \psi$,
iff $\mathfrak{A} \models \psi[a]$, for all $a \in |\mathfrak{A}|$ and ψ such that $\Gamma \vdash \psi$,
iff $\mathfrak{A} \not\models \neg\psi[a]$, for all $a \in |\mathfrak{A}|$ and ψ such that $\Gamma \vdash \psi$,
iff $\mathfrak{A} \not\models \psi[a]$, for all $a \in |\mathfrak{A}|$ and ψ such that $\Gamma \vdash \neg\psi$,
iff $\widehat{\psi} = \mathbf{0}$, for all ψ such that $\Gamma \vdash \neg\psi$ (here $\widehat{\psi}, \mathbf{0} \in \mathbf{M}_0^{\mathfrak{A}}$)

and

$\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \varphi[a]$, for all $a \in |\mathfrak{A}|$,
iff $\mathfrak{A} \not\models \neg\varphi[a]$, for all $a \in |\mathfrak{A}|$,
iff $\widehat{\neg\varphi} = \mathbf{0}$ (here $\widehat{\neg\varphi}, \mathbf{0} \in \mathbf{M}_0^{\mathfrak{A}}$).

Definition 2.4.12 (cf. [7, p. 130])**.** *An **interpretation** of an MBA $\mathbf{A}$ is an MBA-homomorphism of $\mathbf{A}$ into a model.*

Definition 2.4.13. *An element $p \in \mathbf{A}$ is **false** in an interpretation f if $f(p) = \mathbf{0}$. An element $p \in \mathbf{A}$ is **universally invalid** if it is false in every interpretation.*

So the zero element $\mathbf{0} \in \mathbf{A}$ is universally invalid (since every MBA-homomorphism preserves all constants). In particular, for every MBA-logic $(\mathbf{A}, \mathbf{I})$, if $p_0 \in \mathbf{I}$ then $[p_0]$ is false in every interpretation of the MBA $\mathbf{A}/\mathbf{I}$ and so $[p_0]$ is universally invalid.

Definition 2.4.14. *An MBA $\mathbf{A}$ is **semantically complete** if $\mathbf{0}$ is the only universally invalid element, in other words, if $p \in \mathbf{A}$ is universally invalid, then $p = \mathbf{0}$.*

Suppose $\mathbf{A}$ is an MBA and $p \in \mathbf{A}$ is non-zero. Then, by Corollary 2.4.10, there is an interpretation f of $\mathbf{A}$ such that $f(p) \neq \mathbf{0}$. Hence p is not universally invalid. Thus every MBA is semantically complete. So suppose

$(\mathbf{A},\mathbf{I})$ is an MBA-logic. If $p_0 \in \mathbf{A}$ and $[p_0]$ is universally invalid, then $[p_0] = \mathbf{0}$ $(\in \mathbf{A}/\mathbf{I})$. Hence $p_0 \in \mathbf{I}$, i.e. p_0 is refutable.

It is worthwhile noticing that the MBA's $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$, $x \in W_{\mathbf{A}}$, in Theorem 2.4.9 are subdirectly irreducible. Firstly, the MBA's $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$, $x \in W_{\mathbf{A}}$, are not trivial in both cases of the theorem. Secondly, the MBA's $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$ in Case 1 of the theorem may be considered as monadic algebras because $E^x_{\mathbf{A}} = \mathbf{1}$. Moreover, the quantifiers $\langle R^x_{\mathbf{A}}\rangle$ of these algebras are simple. Therefore they are simple in terms of universal algebra. Hence the MBA's in Case 1 are subdirectly irreducible. Thirdly, it follows from the next more general lemma suggested to me by R. Goldblatt (also cf. [2, Lemma 4.1]) that the MBA's $(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}}\rangle)$ in Case 2 of the theorem are subdirectly irreducible.

Lemma 2.4.15 (R. Goldblatt)**.** *Let $\mathcal{F} = (W, R, E)$ be a marked directed graph such that $x \in W$ and for every $y \in W$ there exists $n \geq 0$ with $xR^n y$. Let $\mathbf{A}$ be any subalgebra of the complex algebra $\mathbf{P}_{\mathcal{F}} = (\mathcal{P}(W), \cap, \cup, -, \mathbf{0}, \mathbf{1}, E, \langle R\rangle)$ such that $\{x\} \in \mathbf{A}$. Then $\mathbf{A}$ is subdirectly irreducible.*

Proof. By Theorem 2.4.4, it suffices to prove that there is a smallest non-identity congruence on $\mathbf{A}$. Let τ_x be the smallest congruence on $\mathbf{A}$ containing the pair $\langle \{x\}, \emptyset\rangle$. This exists as $\{x\} \in \mathbf{A}$. Since $\{x\} \neq \emptyset$, we have that the congruence τ_x is not the identity.

Let $\sim$ be any congruence on $\mathbf{A}$ not equal to the identity. Hence there are $X, Y \in \mathbf{A}$ such that $X \sim Y$ and $X \neq Y$. Therefore there is $y \in X + Y$ (symmetric difference of X and Y). Moreover, since $y \in W$, there is $n \geq 0$ with $xR^n y$. So $x \in \langle R\rangle^n\{y\} \subseteq \langle R\rangle^n(X+Y)$. Thus $\{x\} = \{x\} \cap \langle R\rangle^n(X+Y) \sim \{x\} \cap \langle R\rangle^n(Y+Y) = \{x\} \cap \langle R\rangle^n\emptyset = \{x\} \cap \emptyset = \emptyset$. Hence $\{x\} \sim \emptyset$. Therefore $\tau_x \subseteq \sim$ because τ_x is the smallest congruence containing the pair $\langle \{x\}, \emptyset\rangle$.

Thus τ_x is the smallest non-identity congruence on $\mathbf{A}$. □

So, it follows from Theorem 2.4.9 that every MBA is isomorphic to a subdirect product of subdirectly irreducible MBA's, which is in accordance with Birkhoff's theorem [3, Theorem 8.6] known for algebras in general.

Theorem 2.4.16 (Birkhoff)**.** *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

Chapter 3

MBA-varieties

This chapter is concerned with MBA-varieties. In Section 3.1 some well-known definitions and theorems from universal algebra are given and a theory of bounded morphisms is developed. In Section 3.2 we prove that every MBA-variety is generated by its finite special members. In Section 3.3 we show that actually every MBA-variety is generated by at most three special members (not necessarily finite). In Section 3.4 each MBA-variety is equationally characterized.

3.1 Varieties and bounded morphisms

This section may be considered as consisting of two parts. In the first part we give some well-known definitions and theorems from universal algebra (see [3]). Moreover, we discuss some consequences from Chapter 2. In the second part we define special MBA's of three types (Type I, Type II, vacuous) which are originated from the second representation theorem. In addition, adapting R. Goldblatt's frame homomorphisms [6], we develop a theory of bounded morphisms which allows us to work with morphisms of the complex algebras of bounded graphs.

We are going to give some well-known definitions and theorems from universal algebra [3].

Definition 3.1.1. *Define the following operators mapping classes of algebras to classes of algebras (all of the same type):*

- $\mathbf{A} \in I(K)$ *iff* $\mathbf{A}$ *is isomorphic to some member of* K*;*
- $\mathbf{A} \in S(K)$ *iff* $\mathbf{A}$ *is a subalgebra of some member of* K*;*
- $\mathbf{A} \in H(K)$ *iff* $\mathbf{A}$ *is a homomorphic image of some member of* K*;*
- $\mathbf{A} \in P(K)$ *iff* $\mathbf{A}$ *is a direct product of a nonempty family of algebras in* K*.*

Definition 3.1.2. *A nonempty class* K *of algebras of type* $\mathscr{F}$ *is called a* ***variety*** *if it is closed under subalgebras, homomorphic images, and direct products.*

Recall that the type of MBA's is $\{\wedge, \vee, ', \mathbf{0}, \mathbf{1}, E, \exists\}$.

As the intersection of a class of varieties of type $\mathscr{F}$ is again a variety, and as all algebras of type $\mathscr{F}$ form a variety, we can conlude that for every class K of algebras of the same type there is a smallest variety containing K.

Definition 3.1.3. *If* K *is a class of algebras of the same type let* $V(K)$ *denote the smallest variety containing* K*. We say that* $V(K)$ *is the variety generated by* K*. If* $K = \{\mathbf{A}_0, \ldots, \mathbf{A}_{n-1}\}$ *we write simply* $V(\mathbf{A}_0, \ldots, \mathbf{A}_{n-1})$*.*

Theorem 3.1.4 (Tarski)**.** $V = HSP$*.*

Proof. Let us give a sketch of the proof (for complete proof see [3, Theorem 9.5]). Using the properties of the operators H,S and P, it is possible to prove that $HSP(K)$ is a variety (for any class of algebras K). Let $\mathbf{A} \in V(K)$. Then $A \in V'$ for every variety V' with $K \subseteq V'$. In particular, $A \in HSP(K)$. Conversely, let $\mathbf{A} \in HSP(K)$. Then $\mathbf{A}$ is a homomorphic image of $\mathbf{A}'$ where $\mathbf{A}'$ is a subalgebra of the direct product $\prod_{i \in I} \mathbf{A}_i$ for some $\{\mathbf{A}_i \mid i \in I\} \subseteq K$. Therefore $A \in \bigcap\{V' \text{ is a variety } \mid K \subseteq V'\} = V(K)$. □

Definition 3.1.5. *Let X be a set of (distinct) objects called variables. Let $\mathscr{F}$ be a type of algebras and $\mathscr{F}_n$ is the subset of n-ary function symbols in $\mathscr{F}$. The set $T(X)$ of* ***terms*** *of type $\mathscr{F}$ over X is the smallest set such that*

- $X \cup \mathscr{F}_0 \subseteq T(X)$;
- *if $p_0, \ldots, p_{n-1} \in T(X)$ and $f \in \mathscr{F}_n$, then the "string" $f(p_0, \ldots, p_{n-1}) \in T(X)$.*

Definition 3.1.6. *Given a term $p(x_0, \ldots, x_{n-1})$ of type $\mathscr{F}$ over some set X and given an algebra $\mathbf{A}$ of type $\mathscr{F}$ we define a mapping $p^{\mathbf{A}} : A^n \to A$ as follows:*

- *if p is a variable x_i, then*

$$p^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = a_i$$

  for $a_0, \ldots, a_{n-1} \in A$, i.e. $p^{\mathbf{A}}$ is the ith projection;

- *if p is of the form $f(p_0(x_0. \ldots, x_{n-1}), \ldots, p_{k-1}(x_0, \ldots, x_{n-1}))$, where $f \in \mathscr{F}_k$, then*

$$p^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = f^{\mathbf{A}}(p_0^{\mathbf{A}}(a_0. \ldots, a_{n-1}), \ldots, p_{k-1}^{\mathbf{A}}(a_0, \ldots, a_{n-1})).$$

$p^{\mathbf{A}}$ is the ***term function*** *on $\mathbf{A}$ corresponding to the term p.*

Definition 3.1.7. *An* ***identity*** *of type $\mathscr{F}$ over X is an expression of the form*

$$p \approx q$$

where $p, q \in T(X)$. Let $Id(X)$ be the set identities of type $\mathscr{F}$ over X. An algebra $\mathbf{A}$ of type $\mathscr{F}$ satisfies an identity

$$p(x_0, \ldots, x_{n-1}) \approx q(x_0, \ldots, x_{n-1}),$$

(or the identity is true in $\mathbf{A}$, or holds in $\mathbf{A}$), abbreviated by

$$\mathbf{A} \models p(x_0, \ldots, x_{n-1}) \approx q(x_0, \ldots, x_{n-1}),$$

or more briefly

$$\mathbf{A} \models p \approx q,$$

if for every choice $a_0, \ldots, a_{n-1} \in A$ *we have*

$$p^{\mathbf{A}}(a_0, \ldots, a_{n-1}) = q^{\mathbf{A}}(a_0, \ldots, a_{n-1}).$$

A class K *of algebras satisfies* $p \approx q$*, written*

$$K \models p \approx q,$$

if each member of K *satisfies* $p \approx q$*.*

If Σ *is a set of identities, we say* K *satisfies* Σ*, written* $K \models \Sigma$*, if* $K \models p \approx q$ *for each* $p \approx q \in \Sigma$*.*

Given K *and* X *let*

$$Id_K(X) = \{p \approx q \in Id(X) \mid K \models p \approx q\}.$$

We use the symbol $\not\models$ *for "does not satisfy."*

Lemma 3.1.8. *For any class* K *of type* $\mathscr{F}$ *all of the classes* K*,* $I(K)$*,* $S(K)$*,* $H(K)$*,* $P(K)$ *and* $V(K)$ *satisfy the same identities over any set of variables* X*.*

Proof. By properties of isomorphisms, subalgebras, homomorphisms, direct products (for complete proof see [3, Lemma 11.3]). □

Definition 3.1.9. *Let* Σ *be a set of identities of type* $\mathscr{F}$*, and define* $Mod(\Sigma)$ *to be the class of algebras* $\mathbf{A}$ *satisfying* Σ*. A class* K *of algebras is an* ***equational class*** *if there is a set of identities* Σ *such that* $K = Mod(\Sigma)$*. In this case we say that* K *is defined, or axiomatized, by* Σ*.*

Theorem 3.1.10 (Birkhoff)**.** *If* V *is a variety and* X *is an infinite set of variables, then* $V = Mod(Id_V(X))$*.*

Proof. Due to its complexity even the sketch of the proof is not given here (for proof see [3, Lemma 11.8]). □

Theorem 3.1.11 (Birkhoff)**.** *K is an equational class iff K is a variety.*

Proof. Part $\Rightarrow$. Suppose $K = Mod(\Sigma)$. Then $V(K) \models \Sigma$ (by Lemma 3.1.8). Therefore $V(K) \subseteq Mod(\Sigma)$. So $V(K) = K$, i.e. K is a variety. Part $\Leftarrow$ follows from Theorem 3.1.10. □

Definition 3.1.12 ([3, p. 93])**.** *Let X be a set of variables and Σ a set of identities of type $\mathscr{F}$ with variables from X. For $p, q \in T(X)$ we write*

$$\Sigma \models p \approx q$$

(read:"Σ yields $p \approx q$") if, given any algebra $\mathbf{A}$,

$$\mathbf{A} \models \Sigma \textit{ implies } \mathbf{A} \models p \approx q.$$

Definition 3.1.13 ([3, p. 227])**.** *Let X be a set of variables and K a class of algebras. We say that $Id_K(X)$ is **finitely based** if there is a finite subset Σ of $Id_K(X)$ such that*

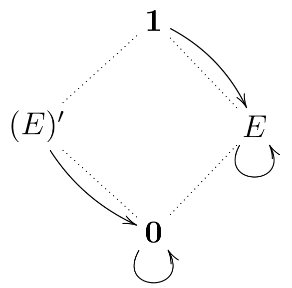
$$\Sigma \models Id_K(X).$$

Now we are in a position to discuss some relations among certain classes of MBA's. Define the following classes:

- Models is the class of all models,
- FunctionalMBA is the class of all functional MBA's,
- MBA is the variety of all MBA's.

So Models $\subseteq$ FunctionalMBA $\subseteq$ MBA and V(Models) $\subseteq V$(FunctionalMBA) $\subseteq$ MBA.

We are going to show that Models $\neq$ MBA. Let $\mathcal{F}_0 = (W_0, R_0, E_0)$, where $W_0 = \{a\}$, $E_0 = \{a\}$, $R_0 = \{\langle a, a\rangle\}$ (see Example 2.2.10(2)), and $\mathcal{F}_1 = (W_1, R_1, E_1)$, where $W_1 = \{b\}$, $E_1 = \emptyset$ and $R_1 = \emptyset$ (see Example 2.2.10(1)). It is easy to see that $\mathcal{F}_0$ and $\mathcal{F}_1$ are bounded graphs and so the complex algebras $\mathbf{P}_{\mathcal{F}_0}$ and $\mathbf{P}_{\mathcal{F}_1}$ are (basic) MBA's. Since the direct product $\mathbf{A} = \mathbf{P}_{\mathcal{F}_0} \times \mathbf{P}_{\mathcal{F}_1}$ is not a basic MBA, we have $\mathsf{Models} \neq \mathsf{MBA}$ (by Theorem 2.3.20). In picture $\mathbf{A}$ looks as follows (the dotted lines denote the usual partial order on $\mathbf{A}$ and the continuous lines denote the quantifier of $\mathbf{A}$):



where $\mathbf{0} = \langle \emptyset, \emptyset \rangle$, $E = \langle \{a\}, \emptyset \rangle$, $E' = \langle \emptyset, \{b\} \rangle$, $\mathbf{1} = \langle \{a\}, \{b\} \rangle$.

Now we are going to show that $\mathsf{FunctionalMBA} \neq \mathsf{MBA}$. Firstly, in any $\mathbf{B}$-valued functional MBA with domain (X, X_E) and designated function E, if $X_E = \emptyset$, then $E = \mathbf{0}$, and if $X_E \neq \emptyset$, then $\exists E = \mathbf{1}$. Secondly, consider the direct product $\mathbf{A}$ again. If $\mathbf{A}$ were isomorphic to some functional MBA, then we would have in $\mathbf{A}$ that either $E = \mathbf{0}$ or $\exists E = \mathbf{1}$. But $E \neq \mathbf{0}$ and $\exists E \neq \mathbf{1}$ (see the picture above). So $\mathbf{A}$ is not isomorphic to any functional MBA. Thus $\mathsf{FunctionalMBA} \neq \mathsf{MBA}$.

Moreover, the MBA $\mathbf{A}$ shows that $I(\mathsf{Models})$ and $I(\mathsf{FunctionalMBA})$ are not closed under direct products. So both $I(\mathsf{Models})$ and $I(\mathsf{FunctionalMBA})$ are not varieties.

However, by Corollary 2.4.10, every MBA is isomorphic to a subdirect product of models. Therefore

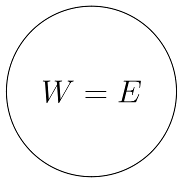
$$V(\mathsf{Models}) = \mathsf{MBA}$$

because every variety is closed under subalgebras, homomorphic images, and direct products. Thus

$$V(\mathsf{Models}) = V(\mathsf{FunctionalMBA}) = \mathsf{MBA}.$$

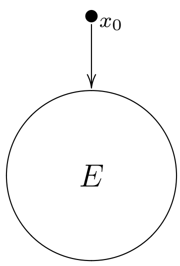
We are now about to introduce some special bounded graphs which are originated from Theorem 2.4.9.

Definition 3.1.14. *A marked directed graph $\mathcal{F} = (W, R, E)$ with $E = W \neq \emptyset$ and $R = W \times W$ is of **Type I**. Diagrammatically:*



Example 3.1.15. *The bounded graphs in Example 2.2.10(2,3) are of Type I.*

Definition 3.1.16. *A marked directed graph $\mathcal{F} = (W, R, E)$ with $W = \{x_0\} \cup E$, $x_0 \notin E \neq \emptyset$ and $R = \{\langle x_0, y\rangle \mid y \in E\} \cup (E \times E)$ is of **Type II**. Diagrammatically:*



Example 3.1.17. *The bounded graph in Example 2.2.10(4) is of Type II.*

Definition 3.1.18. *A marked directed graph $\mathcal{F} = (W, R, E)$ with $W = \{x_0\}$, $R = \emptyset$ and $E = \emptyset$ is **vacuous**. Diagrammatically it is just one point which is not related to itself.*

Suppose $\mathcal{F} = (W, R, E)$ is a marked directed graph of Type I or of Type II or is vacuous. Then $\mathcal{F}$ is actually a bounded graph. Therefore the complex algebra $\mathbf{P}_{\mathcal{F}}$ is an MBA (by Lemma 2.2.11). We can call $\mathcal{F}$ a **bounded graph of Type I**, **of Type II**, or a **vacuous bounded graph**, respectively.

The marked directed graphs $(W^x_{\mathbf{A}}, R^x_{\mathbf{A}}, E^x_{\mathbf{A}})$ in Case 1 of Theorem 2.4.9 are of Type I, the marked directed graphs $(W^x_{\mathbf{A}}, R^x_{\mathbf{A}}, E^x_{\mathbf{A}})$ with $E^x_{\mathbf{A}} \neq \emptyset$ in

Case 2 of Theorem 2.4.9 are of Type II, and the marked directed graphs $(W^x_{\mathbf{A}}, R^x_{\mathbf{A}}, E^x_{\mathbf{A}})$ with $E^x_{\mathbf{A}} = \emptyset$ in Case 2 of Theorem 2.4.9 are vacuous.

Definition 3.1.19. *An MBA $(\mathbf{A}, E, \exists)$ is called **special** if it is equal to the complex algebra $\mathbf{P}_{\mathcal{F}}$, where $\mathcal{F} = (W, R, E)$ is a bounded graph of Type I or Type II or is a vacuous bounded graph. Sometimes we specify by saying "**special MBA of Type I**", "**special MBA of Type II**" or "**vacuous MBA**", respectively.*

Definition 3.1.20. *Every MBA-subalgebra of a special MBA is called a **subspecial MBA**.*

The MBA's $\{(\mathbf{P}^x_{\mathbf{A}}, E^x_{\mathbf{A}}, \langle R^x_{\mathbf{A}} \rangle) \mid x \in W_{\mathbf{A}}\}$ in Theorem 2.4.9 are subspecial, and so we obtain the second corollary of the theorem.

Corollary 3.1.21. *Every MBA is isomorphic to a subdirect product of subspecial MBA's.*

From it we can get the next theorem.

Theorem 3.1.22. *Every variety of MBA's is generated by its subspecial members.*

Proof. Let V be a variety of MBA's and $\mathbf{A} \in V$. Then $\mathbf{A}$ is isomorphic to a subdirect product of subspecial MBA's (by Corollary 3.1.21). Since the product is subdirect and V is a variety, we conclude that each of these subspecial MBA's belongs to V. □

The following definitions and results are adaptations of R. Goldblatt's frame homomorphisms [6, Section 1.5].

Definition 3.1.23 (cf. [6, Definition 1.5.1]). *Let $\mathcal{F} = (W, R, E)$ and $\mathcal{F}' = (W', R', E')$ be marked directed graphs. A **bounded morphism** $f : \mathcal{F} \to \mathcal{F}'$ is a function $f : W \to W'$ such that*

- *for every $u, v \in W$, uRv implies $f(u)R'f(v)$,*
- *for every $u \in W$ and for every $x \in W'$, $f(u)R'x$ implies that there is $v \in W$ such that uRv and $f(v) = x$,*

- *for every* $u \in W$, $u \in E$ *iff* $f(u) \in E'$.

Theorem 3.1.24 (cf. [6, Theorem 1.5.2])**.** *Suppose* $f : \mathcal{F} \to \mathcal{F}'$ *is a bounded morphism. Then, for every* $X, Y \in \mathcal{P}(W')$,

1. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$,
2. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$,
3. $f^{-1}(-X) = -f^{-1}(X)$ *(or, more precisely,* $f^{-1}(W' - X) = W - f^{-1}(X)$*)*,
4. $f^{-1}(\langle R' \rangle X) = \langle R \rangle f^{-1}(X)$,
5. $f^{-1}(E') = E$,

where $f^{-1}(X) = \{u \in W \mid f(u) \in X\}$ *for every* $X \in \mathcal{P}(W')$.

Proof. 1. For every $x \in W$, we have $x \in f^{-1}(X \cap Y)$ iff $f(x) \in X \cap Y$ iff both $f(x) \in X$ and $f(x) \in Y$ iff both $x \in f^{-1}(X)$ and $x \in f^{-1}(Y)$ iff $x \in f^{-1}(X) \cap f^{-1}(Y)$.

2. For every $x \in W$, we have $x \in f^{-1}(X \cup Y)$ iff $f(x) \in X \cup Y$ iff either $f(x) \in X$ or $f(x) \in Y$ iff either $x \in f^{-1}(X)$ or $x \in f^{-1}(Y)$ iff $x \in f^{-1}(X) \cup f^{-1}(Y)$.

3. For every $x \in W$, we have $x \in f^{-1}(W' - X)$ iff $f(x) \in W' - X$ iff $f(x) \in W'$ and $f(x) \notin X$ iff $x \in f^{-1}(W')$ and $x \notin f^{-1}(X)$ iff $x \in W$ and $x \notin f^{-1}(X)$ (since $f^{-1}(W') = W$) iff $x \in W - f^{-1}(X)$.

4. For $\subseteq$, suppose $u \in f^{-1}(\langle R' \rangle X)$ ($X \in \mathcal{P}(W')$)). Hence $f(u) \in \langle R' \rangle X$. Then $f(u)R'x$ for some $x \in X$. Therefore there is $v \in W$ such that uRv and $f(v) = x$ (since f is a bounded morphism). Then $v \in f^{-1}(X)$, and so $u \in \langle R \rangle f^{-1}(X)$. For the other direction, suppose $u \in \langle R \rangle f^{-1}(X)$. Then uRv for some $v \in f^{-1}(X)$. Hence $f(u)R'f(v)$ (since f is a bounded morphism). Since $v \in f^{-1}(X)$, we get $f(v) \in X$. Therefore $f(u) \in \langle R' \rangle X$. So $u \in f^{-1}(\langle R' \rangle X)$.

5. Suppose $u \in W$. Then we have $u \in f^{-1}(E')$ iff $f(u) \in E'$ iff $u \in E$. So $f^{-1}(E') = E$. □

Definition 3.1.25 (cf. [6, Definition 1.5.8]). *If $f : \mathcal{F} \to \mathcal{F}'$ is a bounded morphism, then the mapping*

$$h_f : \mathcal{P}(W') \to \mathcal{P}(W)$$

is defined by

$$h_f(X) = f^{-1}(X),$$

for every $X \in \mathcal{P}(W')$.

It follows from the previous lemma that

- $h_f(X \cap Y) = h_f(X) \cap h_f(Y)$,
- $h_f(X \cup Y) = h_f(X) \cup h_f(Y)$,
- $h_f(-X) = -h_f(X)$ (or, more precisely, $h_f(W' - X) = W - h_f(X)$),
- $h_f(\langle R' \rangle X) = \langle R \rangle h_f(X)$,
- $h_f(E') = E$.

The following lemma allows us to find relations among special MBA's.

Theorem 3.1.26 (cf. [6, Theorem 1.5.9]). *Suppose $\mathcal{F} = (W, R, E)$ and $\mathcal{F}' = (W', R', E')$ are marked directed graphs and $f : \mathcal{F} \to \mathcal{F}'$ is a bounded morphism.*

1. *If f is surjective (i.e. onto), then h_f is injective (i.e. one-to-one).*
2. *If f is injective, then h_f is surjective.*
3. *If f is bijective (i.e. both onto and one-to-one), then h_f is bijective.*

Proof. 1. Suppose $X, Y \in \mathcal{P}(W')$ and $X \neq Y$. Hence there is $x_0 \in W'$ such that either $x_0 \in X \,\&\, x_0 \notin Y$ or $x_0 \notin X \,\&\, x_0 \in Y$. Since $x_0 \in W'$ and f is onto, $x_0 = f(u_0)$ for some $u_0 \in W$. If $x_0 \in X$ and $x_0 \notin Y$, then $u_0 \in f^{-1}(X)$ and $u_0 \notin f^{-1}(Y)$; and so $f^{-1}(X) \neq f^{-1}(Y)$, i.e. $h_f(X) \neq h_f(Y)$. Similarly for the case $x_0 \notin X$ and $x_0 \in Y$.

2. Suppose $U \in \mathcal{P}(W)$. Then $U \subseteq W$ and $f(U) \subseteq W'$. Therefore $h_f(f(U)) = f^{-1}(f(U))$. To be proved that $f^{-1}(f(U)) = U$. Obviously, $f^{-1}(f(U)) \supseteq U$. For $\subseteq$, suppose $u \in f^{-1}(f(U))$. Hence $f(u) \in f(U)$. Then $f(u) = f(v)$ for some $v \in U$. So $u = v$ (since f is one-to-one). Thus $u \in U$.

3. Follows from (1,2). □

Corollary 3.1.27. *Suppose $\mathcal{F} = (W, R, E)$ and $\mathcal{F}' = (W', R', E')$ are bounded graphs and $f : \mathcal{F} \to \mathcal{F}'$ is a bounded morphism. Then the mapping $h_f : \mathcal{P}(W') \to \mathcal{P}(W)$ is an MBA-homomorphism from $\mathbf{P}_{\mathcal{F}'}$ to $\mathbf{P}_{\mathcal{F}}$. Moreover,*

- *if f is surjective, then h_f is an injective MBA-homomorphism from $\mathbf{P}_{\mathcal{F}'}$ to $\mathbf{P}_{\mathcal{F}}$, and hence $\mathbf{P}_{\mathcal{F}'}$ is isomorphic to a subalgebra of $\mathbf{P}_{\mathcal{F}}$;*
- *if f is injective, then h_f is a surjective MBA-homomorphism from $\mathbf{P}_{\mathcal{F}'}$ to $\mathbf{P}_{\mathcal{F}}$;*
- *if f is bijective, then h_f is an MBA-isomorphism.* □

The Lemmas 3.1.28, 3.1.29, 3.1.30 state connections between bounded graphs of the same type, whereas Lemma 3.1.31 between bounded graphs of Type I and Type II.

Lemma 3.1.28. *If $\mathcal{F} = (W, R, E)$ and $\mathcal{F}' = (W', R', E')$ are bounded graphs of Type I and $Card(W) = Card(W')$, then there is a bijective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$.*

Proof. Since $Card(W) = Card(W')$, there is a bijective mapping $f : W \to W'$. From the assumption that $\mathcal{F} = (W, R, E)$ and $\mathcal{F}' = (W', R', E')$ are bounded graphs of Type I it follows that f is a bounded morphism. □

Lemma 3.1.29. *If $\mathcal{F} = (W, R, E)$ and $\mathcal{F}' = (W', R', E')$ are bounded graphs of Type II and $Card(E) = Card(E')$, then there is a bijective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$.*

Proof. Suppose $W = \{x_0\} \cup E$, $x_0 \notin E \neq \emptyset$, $R = \{\langle x_0, y\rangle \mid y \in E\} \cup (E \times E)$ and $W' = \{x'_0\} \cup E'$, $x'_0 \notin E' \neq \emptyset$ and $R' = \{\langle x'_0, y\rangle \mid y \in E'\} \cup (E' \times E')$.

Since $Card(E) = Card(E')$, there is a bijective mapping $f : E \to E'$. Then define $g : W \to W'$ by

$$g(u) = \begin{cases} x'_0, & \text{if } u = x_0 \\ f(u), & \text{if } u \in E \end{cases}$$

for every $u \in W$.

To be proved that g is a bijective bounded morphism. Obviously, g is bijective. Suppose $u, v \in W$ and uRv. Hence either $u = x_0 \,\&\, v \in E$ or $u, v \in E$. In the first case, $g(u) = x'_0$ and $g(v) \in E'$, and in the second, $g(u), g(v) \in E'$. Therefore $g(u)R'g(v)$ in both cases. Now suppose that $u \in W, x \in W'$ and $g(u)R'x$. Since $x \in W'$ and g is a bijection between W and W', there is $v \in W$ with $g(v) = x$. Since $g(u)R'x$, either $g(u) = x'_0 \,\&\, x \in E'$ or $g(u), x \in E'$. In the first case, $u = x_0$ and $v \in E$, and in the second, $u, v \in E$. Therefore uRv in both cases. Finally, by construction of g, $u \in E$ iff $g(u) \in E'$ for every $u \in W$. Thus g is a bijective bounded morphism. □

Lemma 3.1.30. *If $\mathcal{F} = (W, R, E)$ and $\mathcal{F}' = (W', R', E')$ are vacuous bounded graphs, then there is a bijective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$.*

Proof. Obvious. □

Lemma 3.1.31. *Suppose $\mathcal{F} = (W, R, E)$ is a bounded graph of Type I, $\mathcal{F}' = (W', R', E')$ is a bounded graph of Type II and $Card(E) = Card(E')$. Then there is an injective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$.*

Proof. Suppose $E = W \neq \emptyset$, $R = W \times W$ and $W' = \{x'_0\} \cup E'$, $x'_0 \notin E' \neq \emptyset$, $R' = \{\langle x'_0, y\rangle \mid y \in E'\} \cup (E' \times E')$. Since $Card(E) = Card(E')$, there is a bijection $f : E \to E'$.

To be proved that f is an injective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$. Obviously, f is injective. Suppose $u, v \in W$ and uRv. Then $u, v \in E$, and so $f(u), f(v) \in E'$. Therefore $f(u)R'f(v)$ (since $E' \times E' \subseteq R'$). Now suppose $u \in W$, $x \in W'$ and $f(u)R'x$. Then $x \in E'$. Hence there is (only one) $v \in E$ such that $f(v) = x$. So uRv. Finally, for every $u \in W$, $u \in E$ iff $f(u) \in E'$

(since f is a mapping from E to E' and $W = E$). Thus f is an injective bounded morphism. □

3.2 MBA-varieties are generated by their finite special members

The purpose of this section is to prove that every MBA-variety is generated by its finite special members. Let us firstly find a sufficient condition for that. Suppose X is an infinite set of variables. Let V be a variety of algebras (not necessarily of MBA's) and $K \subseteq V$. To be shown that $Id_K(X) = Id_V(X)$ implies that K generates V (i.e. $V(K) = V$). Let $Id_K(X) = Id_V(X)$. Then $Id_{V(K)}(X) = Id_V(X)$ (by Lemma 3.1.8). Hence $Mod(Id_{V(K)}(X)) = Mod(Id_V(X))$, and so $V(K) = V$ (by Theorem 3.1.10). Thus K generates V. But $K \subseteq V$ implies that $Id_V(X) \subseteq Id_K(X)$. Therefore, to prove that K generates V it suffices to show that $Id_K(X) \subseteq Id_V(X)$ (or, equivalently, if $(t_0 \approx t_1) \notin Id_V(X)$, then $(t_0 \approx t_1) \notin Id_K(X)$) . Moreover, since in every Boolean algebra (and so in every MBA)

$$a = b \text{ iff } (a' \vee b) \wedge (a \vee b') = \mathbf{1},$$

it suffices to consider MBA-equations of the form $t(x_0, \ldots, x_{n-1}) = \mathbf{1}$ only. In the theorem below the method of filtration due to E.J. Lemmon [8] is taken for granted. Moreover, the notion of distinguished model from modal logic [5, p. 36] is tacitly used.

Theorem 3.2.1. *Suppose V is a variety of MBA's, $K \subseteq V$ is the subset of all finite special members in V, and $t(x_0, \ldots, x_{n-1})$ is an MBA-term. If*

$$\mathbf{A}_0 \not\models t(x_0, \ldots, x_{n-1}) \approx \mathbf{1}$$

for some $\mathbf{A}_0 \in V$, then

$$\mathbf{A}_1 \not\models t(x_0, \ldots, x_{n-1}) \approx \mathbf{1}$$

for some $\mathbf{A}_1 \in K$.

Proof. Suppose $\mathbf{A}_0 \in V$ and $\mathbf{A}_0$ does not satisfy $t(x_0, \dots, x_{n-1}) \approx \mathbf{1}$. Therefore, by Corollary 3.1.21, there exists $\mathbf{A} \in V$ such that $\mathbf{A}$ does not satisfy $t(x_0, \dots, x_{n-1}) \approx \mathbf{1}$ and one of the following holds:

1. $\mathbf{A}$ is equal to the complex algebra $\mathbf{P}_{\mathcal{F}}$ of some vacuous bounded graph $\mathcal{F}$;

2. $\mathbf{A}$ is an MBA-subalgebra of the complex algebra $\mathbf{P}_{\mathcal{F}}$ of some bounded graph $\mathcal{F}$ of Type I;

3. $\mathbf{A}$ is an MBA-subalgebra of the complex algebra $\mathbf{P}_{\mathcal{F}}$ of some bounded graph $\mathcal{F}$ of Type II.

In (1), since $\mathbf{A}$ is finite and special, we can take for the desired $\mathbf{A}_1$ the MBA $\mathbf{A}$ itself. So it remains to consider only (2) and (3). Therefore we reformulate them by saying that $\mathbf{A}$ is an MBA-subalgebra of $\mathbf{P}_{\mathcal{F}}$ of some bounded graph $\mathcal{F} = (W, R, E)$ such that either

Case 1 $E = W \neq \emptyset$ and $R = W \times W$, or

Case 2 $W = \{x_0\} \cup E$, $x_0 \notin E \neq \emptyset$ and $R = \{\langle x_0, y\rangle \mid y \in E\} \cup (E \times E)$.

Since $\mathbf{A} \not\models t(x_0, \dots, x_{n-1}) \approx \mathbf{1}$, there are $A_0, \dots, A_{n-1} \in \mathbf{A}$ such that $t^{\mathbf{A}}(A_0, \dots, A_{n-1}) \neq \mathbf{1}^{\mathbf{A}}$, i.e.

$$t^{\mathbf{A}}(A_0, \dots, A_{n-1}) \neq W.$$

Note that $A_0, \dots, A_{n-1} \subseteq W$.

We are going apply the method of filtration due to E.J. Lemmon (see [8, Theorem 40]). Let $\{t_0, \dots, t_{r-1}\}$ be the set of all subterms of the term $t(x_0, \dots, x_{n-1})$. Then put

$$B_i = t_i^{\mathbf{A}}(A_0, \dots, A_{n-1}),$$

for each $0 \leq i \leq r-1$,

$$B_r = E^{\mathbf{A}},$$

and

$$S = \{B_0, \dots, B_{r-1}, B_r\}.$$

Thus $B_0, \dots, B_{r-1}, B_r \subseteq W$, $S \subseteq \mathcal{P}(W)$, and $B_i \in \mathbf{A}$ (for every $i \leq r$). Also note that S is finite.

Define a binary relation $\equiv$ on W by

$$x \equiv y \text{ iff } (\forall B \in \mathcal{P}(W))(B \in S \to (x \in B \leftrightarrow y \in B)). \tag{3.2.1}$$

Then $\equiv$ is an equivalence relation on W which partitions W into not more than 2^{r+1} equivalence classes (because there are $r+1$ members in S).

For every $x \in W$, put $\bar{x} = \{y \in W \mid x \equiv y\}$, and for $B \subseteq W$, $\bar{B} = \{\bar{x} \mid x \in B\}$. So $\bar{W}$ and $\mathcal{P}(\bar{W})$ are finite.

For $x \in W$ and $B \subseteq W$, it is easy to verify that

(α) if $x \in B$, then $\bar{x} \in \bar{B}$;

(β) if $\bar{x} \in \bar{B}$ and $B \in S$, then $x \in B$.

Define a binary relation $\bar{R}$ on $\bar{W}$ by

$$\bar{x}\bar{R}\bar{y} \text{ iff there exist } x' \in \bar{x} \text{ and } y' \in \bar{y} \text{ such that } x'Ry'. \tag{3.2.2}$$

So we have a marked directed graph $\bar{\mathcal{F}} = (\bar{W}, \bar{R}, \bar{E})$, where $\bar{E} = \overline{E^{\mathbf{A}}}$ (i.e. $\bar{E} = \{\bar{x} \mid x \in E^{\mathbf{A}}\}$). Hence we get the complex algebra $\mathbf{P}_{\bar{\mathcal{F}}}$ of $\bar{\mathcal{F}}$. ($\mathbf{P}_{\bar{\mathcal{F}}}$ is the desired $\mathbf{A}_1$.)

As in [8, p. 209], we have

(i) $B_i = W$ iff $\bar{B}_i = \bar{W}$,

(ii) $-B_i = B_j$ iff $-\bar{B}_i = \bar{B}_j$,

(iii) $B_i \cap B_j = B_k$ iff $\bar{B}_i \cap \bar{B}_j = \bar{B}_k$,

(iv) $B_i \cup B_j = B_k$ iff $\bar{B}_i \cup \bar{B}_j = \bar{B}_k$,

(v) if $\langle R\rangle B_i = B_j$, then $\langle \bar{R}\rangle \bar{B}_i = \bar{B}_j$, for every $i, j, k \leq r$.

It is possible by induction on the term t to prove that

$$\overline{t^{\mathbf{A}}(A_0, \ldots, A_{n-1})} = t^{\mathbf{P}_{\bar{\mathcal{F}}}}(\bar{A}_0, \ldots, \bar{A}_{n-1}). \tag{3.2.3}$$

Since every term is a subterm of itself, $t^{\mathbf{A}}(A_0, \ldots, A_{n-1}) \in S$. Therefore $\overline{t^{\mathbf{A}}(A_0, \ldots, A_{n-1})} \neq \bar{W}$ (since $t^{\mathbf{A}}(A_0, \ldots, A_{n-1}) \neq W$ and using (i)). Hence $t^{\mathbf{P}_{\bar{\mathcal{F}}}}(\bar{A}_0, \ldots, \bar{A}_{n-1}) \neq \bar{W}$ (by (3.2.3)), or $t^{\mathbf{P}_{\bar{\mathcal{F}}}}(\bar{A}_0, \ldots, \bar{A}_{n-1}) \neq \mathbf{1}^{\mathbf{P}_{\bar{\mathcal{F}}}}$. So the algebra $\mathbf{P}_{\bar{\mathcal{F}}}$ does not satisfy the equation $t(x_0, \ldots, x_{n-1}) \approx \mathbf{1}$.

It remains to prove that $\mathbf{P}_{\bar{\mathcal{F}}}$ is a finite special MBA and that it belongs to V (so $\mathbf{P}_{\bar{\mathcal{F}}} \in K$). To accomplish it we will follow the plan: a) $\mathbf{P}_{\bar{\mathcal{F}}}$ is a finite special MBA, b) there is a surjective bounded morphism f from $\mathcal{F}$ onto $\bar{\mathcal{F}}$, and c) $h_f(\mathbf{P}_{\bar{\mathcal{F}}}) \subseteq \mathbf{A}$. Both in (a) and (b) we will consider two cases (according to the two cases on p. 51), whereas in (c) we will not.

Part (a).

Case 1. Since $E = W$, we have $\bar{E} = \bar{W}$. To be proved that $\bar{R} = \bar{W} \times \bar{W}$. By definition of $\bar{R}$, $\bar{R} \subseteq \bar{W} \times \bar{W}$. For $\supseteq$, assume $\langle \bar{x}, \bar{y} \rangle \in \bar{W} \times \bar{W}$. Then xRy, since $x \in \bar{x}, y \in \bar{y}$ and $R = W \times W$. So $\bar{x}\bar{R}\bar{y}$ (by definition of $\bar{R}$). Thus the marked directed graph $\bar{\mathcal{F}}$ is a bounded graph of Type I. Hence $\mathbf{P}_{\bar{\mathcal{F}}}$ is a finite special MBA.

Case 2. Since $x_0 \neq y$, for every $y \in E$, and $E \in S$, we have $\bar{x}_0 = \{x_0\}$. It follows from $x_0 \notin E$ and $E \in S$ that $\bar{x}_0 \notin \bar{E}$ (by β). Since $E \neq \emptyset$, we have $\bar{E} \neq \emptyset$.

To be proved that $\bar{W} = \{\bar{x}_0\} \cup \bar{E}$. By definition, $\bar{W} \supseteq \{\bar{x}_0\} \cup \bar{E}$. For $\subseteq$, suppose $\bar{x} \in \bar{W}$. If $\bar{x} = \bar{x}_0$, then $\bar{x} \in \bar{x}_0 \cup \bar{E}$. If $\bar{x} \neq \bar{x}_0$, then $x \neq x_0$ (since $\equiv$ is an equivalence relation on W). Hence $x \in E$. So $\bar{x} \in \bar{E}$ (by (α)). Therefore $\bar{x} \in \bar{x}_0 \cup \bar{E}$.

Next to be proved that $\bar{R} = \{\langle \bar{x}_0, \bar{y} \rangle \mid \bar{y} \in \bar{E}\} \cup (\bar{E} \times \bar{E})$. For $\subseteq$, suppose $\bar{x}_1, \bar{x}_2 \in \bar{W}$ and $\langle \bar{x}_1, \bar{x}_2 \rangle \in \bar{R}$. Hence there exist $u \in \bar{x}_1$ and $v \in \bar{x}_2$ such that uRv. Then, by assumption on R, there are two cases:

- $u = x_0$ and $v \in E$. Hence $\bar{u} = \bar{x}_0$ and $\bar{v} \in \bar{E}$. Since $\bar{u} = \bar{x}_1$ and $\bar{v} = \bar{x}_2$, we obtain that $\bar{x}_1 = \bar{x}_0$ and $\bar{x}_2 \in \bar{E}$. So $\langle \bar{x}_1, \bar{x}_2 \rangle \in \{\langle \bar{x}_0, \bar{y} \rangle \mid \bar{y} \in \bar{E}\}$.

Thus $\langle \bar{x}_1, \bar{x}_2 \rangle \in \{\langle \bar{x}_0, \bar{y} \rangle \mid \bar{y} \in \bar{E}\} \cup (\bar{E} \times \bar{E})$.

- $u, v \in E$. Hence $\bar{u} \in \bar{E}$ and $\bar{v} \in \bar{E}$. Since $\bar{u} = \bar{x}_1$ and $\bar{v} = \bar{x}_2$, we obtain that $\bar{x}_1 \in \bar{E}$ and $\bar{x}_2 \in \bar{E}$. So $\langle \bar{x}_1, \bar{x}_2 \rangle \in \bar{E} \times \bar{E}$. Thus $\langle \bar{x}_1, \bar{x}_2 \rangle \in \{\langle \bar{x}_0, \bar{y} \rangle \mid \bar{y} \in \bar{E}\} \cup (\bar{E} \times \bar{E})$.

For $\supseteq$, suppose $\langle \bar{x}_1, \bar{x}_2 \rangle \in \{\langle \bar{x}_0, \bar{y} \rangle \mid \bar{y} \in \bar{E}\} \cup (\bar{E} \times \bar{E})$. Then there are two cases:

- $\langle \bar{x}_1, \bar{x}_2 \rangle \in \{\langle \bar{x}_0, \bar{y} \rangle \mid \bar{y} \in \bar{E}\}$. Hence $\bar{x}_1 = \bar{x}_0$ and $\bar{x}_2 \in \bar{E}$. Since $\bar{x}_0 = \{x_0\}$ and $\bar{x}_1 = \bar{x}_0$, we have $x_1 = x_0$. It follows from $\bar{x}_2 \in \bar{E}$ that $x_2 \in E$ (by (β) using $E \in S$). Since $x_1 = x_0$ and $x_2 \in E$, we get $x_1 R x_2$ (by assumption on R). Therefore $\bar{x}_1 \bar{R} \bar{x}_2$ (by definition of $\bar{R}$).

- $\langle \bar{x}_1, \bar{x}_2 \rangle \in \bar{E} \times \bar{E}$. Hence $x_1, x_2 \in E$ (by (α) using $E \in S$). Then $x_1 R x_2$ (by assumption on R). Therefore $\bar{x}_1 \bar{R} \bar{x}_2$ (by definition of $\bar{R}$).

Thus we have proved the equality.

So the marked directed graph $\bar{\mathcal{F}} = (\bar{W}, \bar{R}, \bar{E})$ is a bounded graph of Type II (or, more explicitly, $\bar{W} = \{\bar{x}_0\} \cup \bar{E}$, $\bar{x}_0 \notin \bar{E} \neq \emptyset$, and $\bar{R} = \{\langle \bar{x}_0, \bar{y} \rangle \mid \bar{y} \in \bar{E}\} \cup (\bar{E} \times \bar{E})$). Hence $\mathbf{P}_{\bar{\mathcal{F}}}$ is a finite special MBA.

Thus $\mathbf{P}_{\bar{\mathcal{F}}}$ is a finite special MBA in both cases.

Part (b).

Now define a mapping $f : W \to \bar{W}$ by

$$f(x) = \bar{x},$$

for every $x \in W$. Obviously, f is surjective. We are going to prove that f is a bounded morphism from $\mathcal{F} = (W, R, E)$ onto $\bar{\mathcal{F}} = (\bar{W}, \bar{R}, \bar{E})$.

Case 1.

- Let $u, v \in W$ and uRv. Then $u \in \bar{u}$ and $v \in \bar{v}$. Therefore $\bar{u} \bar{R} \bar{v}$ (by definition of $\bar{R}$). Thus $f(x) \bar{R} f(y)$.

- Let $u \in W$, $x \in \bar{W}$ and $f(u) \bar{R} x$. Since $x \in \bar{W}$, we have that $x = \bar{v}$ for some $v \in W$. Hence $x = f(v)$. Since $R = W \times W$ and $u, v \in W$, we get uRv.

- For every $u \in W$, $u \in E$ iff $\bar{u} \in \bar{E}$ (since $E \in S$ and using (α, β)) iff $f(u) \in \bar{E}$.

So f is a bounded morphism from $\mathcal{F}$ onto $\bar{\mathcal{F}}$.

Case 2.

- Let $u, v \in W$ and uRv. Then $u \in \bar{u}$ and $v \in \bar{v}$. Therefore $\bar{u}\bar{R}\bar{v}$ (by definition of $\bar{R}$). Thus $f(x)\bar{R}f(y)$.

- Let $u \in W$, $x \in \bar{W}$ and $f(u)\bar{R}x$. Since $x \in \bar{W}$, we have that $x = \bar{v}$ for some $v \in W$. Hence $x = f(v)$. Then $f(u)\bar{R}f(v)$ and $\bar{u}\bar{R}\bar{v}$. Therefore there exist $w_0 \in \bar{u}$ and $w_1 \in \bar{v}$ such that $w_0 R w_1$. Then, by assumption on R, there are two cases:

  - $w_0 = x_0$ and $w_1 \in E$. Hence $\bar{w}_0 = \bar{x}_0$ and $\bar{w}_1 \in \bar{E}$. Since $\bar{w}_0 = \bar{u}$ and $\bar{w}_1 = \bar{v}$, we obtain $\bar{u} = \bar{x}_0$ and $\bar{v} \in \bar{E}$. Since $\bar{x}_0 = \{x_0\}$ and $\bar{u} = \bar{x}_0$, we get $u = x_0$. It follows from $\bar{v} \in \bar{E}$ that $v \in E$ (by (β) using $E \in S$). Since $u = x_0$ and $v \in E$, we have uRv (by assumption on R). Thus uRv and $x = f(v)$.

  - $w_0, w_1 \in E$. Since $w_0 \in \bar{u}$, we have $w_0 \equiv u$. Hence $u \in E$ (since $w_0 \in E$ and $E \in S$). Analogously, $v \in E$. Since $u, v \in E$, we get that uRv (by assumption on R). Thus uRv and $x = f(v)$.

- For any $u \in W$, $u \in E$ iff $\bar{u} \in \bar{E}$ (by (α, β) using $E \in S$) iff $f(u) \in \bar{E}$ (by definition of f).

So f is a bounded morphism from $\mathcal{F}$ onto $\bar{\mathcal{F}}$.

Thus f is a surjective bounded morphism from $\mathcal{F}$ onto $\bar{\mathcal{F}}$ in both cases.

Part (c).

It follows from Part (b) that h_f is an injective MBA-homomorphism from $\mathbf{P}_{\bar{\mathcal{F}}}$ into $\mathbf{P}_{\mathcal{F}}$ (by Corollary 3.1.27), and so $\mathbf{P}_{\bar{\mathcal{F}}}$ is isomorphic to an MBA-subalgebra of $\mathbf{P}_{\mathcal{F}}$. Note that we cannot yet conclude that $\mathbf{P}_{\bar{\mathcal{F}}} \in V$, because $\mathbf{P}_{\mathcal{F}}$ may not belong to V (although $\mathbf{A} \in V$ by assumption). But after we prove that $h_f(\mathbf{P}_{\bar{\mathcal{F}}}) \subseteq \mathbf{A}$, we will be able to obtain that $\mathbf{P}_{\bar{\mathcal{F}}} \in V$.

Since, for every $B \in S$, $h_f(\bar{B}) = f^{-1}(\bar{B}) = \{x \in W \mid f(x) \in \bar{B}\} = \{x \in W \mid \bar{x} \in \bar{B}\} = B$ (because $B \in S$ and using (α, β)), we have

$$h_f(\bar{B}) = B \text{ for every } B \in S. \tag{3.2.4}$$

Let $\mathbf{B}_S$ be the subalgebra generated by $\{\bar{B} \mid B \in S\}$ of the Boolean algebra $\mathcal{P}(\bar{W})$. To be proved that $\{\bar{x}\} \in \mathbf{B}_S$ for every $\bar{x} \in \bar{W}$. Suppose $\bar{x} \in \bar{W}$. We are going to represent $\bar{x}$ as a (finite) Boolean combination of elements in $\{\bar{B} \mid B \in S\}$. Assume that $\bar{y} \in \bar{W}$ and $\bar{x} \neq \bar{y}$. Hence there is $B \in S$ such that either $x \in B$ and $y \notin B$ or $x \notin B$ and $y \in B$. If $x \in B$ and $y \notin B$, then $\bar{x} \in \bar{B}$ and $\bar{y} \notin \bar{B}$ (since $B \in S$ and using (α, β)). If $x \notin B$ and $y \in B$, then $\bar{x} \notin \bar{B}$ and $\bar{y} \in \bar{B}$; and so $\bar{x} \in (-\bar{B})$ and $\bar{y} \notin (-\bar{B})$. Therefore $\bar{x} \in \bar{B}_{\bar{y}}$ and $\bar{y} \notin \bar{B}_{\bar{y}}$, where

$$\bar{B}_{\bar{y}} = \begin{cases} \bar{B}, & \text{if } \bar{x} \in \bar{B} \text{ and } \bar{y} \notin \bar{B} \\ -\bar{B}, & \text{if } \bar{x} \notin \bar{B} \text{ and } \bar{y} \in \bar{B}. \end{cases}$$

Note that $\bar{B}_{\bar{y}} \in \mathbf{B}_S$. Since $\bar{W}$ is finite, the collection $\{\bar{B}_{\bar{y}} \mid \bar{x} \neq \bar{y}\}$ is finite and $\bigcap\{\bar{B}_{\bar{y}} \mid \bar{x} \neq \bar{y}\} \in \mathbf{B}_S$. But $\{\bar{x}\} = \bigcap\{\bar{B}_{\bar{y}} \mid \bar{x} \neq \bar{y}\}$. Thus $\{\bar{x}\} \in \mathbf{B}_S$. (Compare with the notion of distinguished model in modal logic in [5, p. 36].)

So $\mathbf{B}_S = \mathcal{P}(\bar{W})$. Hence every element in $\mathcal{P}(\bar{W})$ is a (finite) Boolean combination of elements in $\{\bar{B} \mid B \in S\}$. Therefore $h_f(\mathcal{P}(\bar{W})) \subseteq \mathbf{A}$ by (3.2.4) using the fact that h_f preserves finite Boolean combinations and $\mathbf{A}$ is closed under finite Boolean combinations.

Since $h_f : \mathbf{P}_{\bar{\mathcal{F}}} \to \mathbf{P}_{\mathcal{F}}$ is an injective MBA-homomorphism and $h_f(\mathcal{P}(\bar{W})) \subseteq \mathbf{A}$, we obtain that the complex algebra $\mathbf{P}_{\bar{\mathcal{F}}}$ is isomorphic to an MBA-subalgebra (namely, to $h_f(\mathbf{P}_{\bar{\mathcal{F}}})$) of the MBA $\mathbf{A}$. Thus $\mathbf{P}_{\bar{\mathcal{F}}} \in V$ (since $\mathbf{A} \in V$ and V is closed under subalgebras and homomorphic images).

Finally, since $\mathbf{P}_{\bar{\mathcal{F}}}$ is a finite special MBA, we get $\mathbf{P}_{\bar{\mathcal{F}}} \in K$. □

Corollary 3.2.2. *Every variety of MBA's is generated by its finite special members.*

The following corollary will be useful in the next section.

Corollary 3.2.3. *If $\mathbf{A}_0$ in the theorem is equal to $\mathbf{P}_{\mathcal{F}}$, where $\mathcal{F}$ is a bounded graph of Type I or Type II or is a vacuous bounded graph, then $\mathbf{A}_1 = \mathbf{P}_{\mathcal{F}'}$, where $\mathcal{F}'$ is a finite bounded graph of Type I or Type II or is a vacuous bounded graph, respectively. In other words, if an equation is not satisfied by a special MBA, then it is not satisfied by some finite special MBA of the same type (see Definition 3.1.19).*

3.3 Characterization of MBA-varieties in terms of their generators

In this section we sharpen (in terms of numbers of generators) Corollary 3.2.2. Firstly, bounded graphs $\mathcal{F}_n$,$\mathcal{F}_m^\infty$, $\mathcal{F}_0^\infty$ on subsets of $\{\infty, 0, 1, 2, \dots\}$ are introduced. Secondly, we find relations among the complex algebras $\mathbf{P}_{\mathcal{F}_n}$,$\mathbf{P}_{\mathcal{F}_m^\infty}$, $\mathbf{P}_{\mathcal{F}_0^\infty}$. Finally, we prove that every MBA-variety is generated by some subset of $\{\mathbf{P}_{\mathcal{F}_n}, \mathbf{P}_{\mathcal{F}_m^\infty}, \mathbf{P}_{\mathcal{F}_0^\infty}\}$ (for some $1 \le n \le \omega$ and $1 \le m \le \omega$). As a consequence, we obtain that there are countably many varieties of MBA's.

Let $0 = \emptyset$, $n = \{0, 1, \dots, n-1\}$, and $\omega = \{0, 1, 2, \dots\}$. Suppose ∞ is an entity not in ω.

Definition 3.3.1. *For every $1 \le n \le \omega$, the marked directed graph $\mathcal{F}_n$ is defined by $\mathcal{F}_n = (W, R, E)$ where $W = n$, $R = n \times n$ and $E = n$.*

Definition 3.3.2. *For every $0 \le n \le \omega$, the marked directed graph $\mathcal{F}_n^\infty$ is defined by $\mathcal{F}_n^\infty = (W, R, E)$ where $W = \{\infty\} \cup n$, $R = \{\langle \infty, x\rangle \mid x \in n\} \cup (n \times n)$ and $E = n$.*

Note that $\mathcal{F}_n$, for $1 \le n \le \omega$, is a bounded graph of Type I; $\mathcal{F}_n^\infty$, for $1 \le n \le \omega$, is a bounded graph of Type II; and $\mathcal{F}_0^\infty$ is a vacuous bounded graph.

Definition 3.3.3. *For every $1 \le n \le \omega$, define the MBA $\mathbf{P}_n$ by $\mathbf{P}_n = \mathbf{P}_{\mathcal{F}_n}$.*

Definition 3.3.4. *For every $0 \le n \le \omega$, define the MBA $\mathbf{P}_n^\infty$ by $\mathbf{P}_n^\infty = \mathbf{P}_{\mathcal{F}_n^\infty}$.*

So $\mathbf{P}_n$, $\mathbf{P}_n^\infty$ and $\mathbf{P}_0^\infty$ are finite special MBA's for $1 \leq n < \omega$.

Lemma 3.3.5. *Suppose* $\mathbf{A}$ *is a finite MBA.*

1. *If* $\mathbf{A} = \mathbf{P}_\mathcal{F}$ *for some bounded graph* $\mathcal{F}$ *of Type I, then* $\mathbf{A}$ *is isomorphic to* $\mathbf{P}_n$ *for some* $1 \leq n < \omega$;
2. *If* $\mathbf{A} = \mathbf{P}_\mathcal{F}$ *for some bounded graph* $\mathcal{F}$ *of Type II, then* $\mathbf{A}$ *is isomorphic to* $\mathbf{P}_n^\infty$ *for some* $1 \leq n < \omega$;
3. *If* $\mathbf{A} = \mathbf{P}_\mathcal{F}$ *for some vacuous bounded graph* $\mathcal{F}$, *then* $\mathbf{A}$ *is isomorphic to* $\mathbf{P}_0^\infty$.

Proof. 1. Suppose $\mathbf{A} = \mathbf{P}_\mathcal{F}$ where $\mathcal{F} = (W, R, E)$, $E = W \neq \emptyset$, $R = W \times W$. Since $\mathbf{A}$ is finite, we have $1 \leq Card(W) < \omega$ (say $Card(W) = n$). Therefore there exists a bijective bounded morphism between $\mathcal{F}$ and $\mathcal{F}_n$ (by Lemma 3.1.28). Then the MBA's $\mathbf{A}$ and $\mathbf{P}_n$ are isomorphic (by Corollary 3.1.27).

2. Suppose $\mathbf{A} = \mathbf{P}_\mathcal{F}$ where $\mathcal{F} = (W, R, E)$, $W = \{x_0\} \cup E$, $x_0 \notin E \neq \emptyset$ and $R = \{\langle x_0, y\rangle \mid y \in E\} \cup (E \times E)$. Since $\mathbf{A}$ is finite, we have $1 \leq Card(E) < \omega$ (say $Card(E) = n$). Therefore there exists a bijective bounded morphism between $\mathcal{F}$ and $\mathcal{F}_n^\infty$ (by Lemma 3.1.29). Then the MBA's $\mathbf{A}$ and $\mathbf{P}_n^\infty$ are isomorphic (by Corollary 3.1.27).

3. Suppose $\mathbf{A} = \mathbf{P}_\mathcal{F}$ and $\mathcal{F}$ is a vacuous structure. Therefore there is a bijective bounded morphism between $\mathcal{F}$ and $\mathcal{F}_0^\infty$ (by Lemma 3.1.30). Then the MBA's $\mathbf{A}$ and $\mathbf{P}_0^\infty$ are isomorphic (by Corollary 3.1.27). □

Let V be an arbitrary variety of MBA's.

Lemma 3.3.6. *For every* $1 \leq n \leq m \leq \omega$, $\mathbf{P}_n$ *is isomorphic to an MBA-subalgebra of* $\mathbf{P}_m$.

Proof. Since $n \leq m$, we have $n \subseteq m$. Then define a mapping $f : m \to n$ by

$$f(j) = \begin{cases} j, & \text{if } j \in n \\ 0, & \text{if } j \in m - n \end{cases} \tag{3.3.1}$$

for every $j \in m$ (here $m - n$ is set-theoretical difference). Obviously, f is surjective. It follows from the fact that both $\mathcal{F}_n$ and $\mathcal{F}_m$ are bounded graphs of Type I that f is a bounded morphism from $\mathcal{F}_m$ to $\mathcal{F}_n$. So $f : \mathcal{F}_m \to \mathcal{F}_n$ is a surjective bounded morphism. Therefore $h_f : \mathbf{P}_n \to \mathbf{P}_m$ is an injective MBA-homomorphism (by Corollary 3.1.27). Hence $\mathbf{P}_n$ is isomorphic to an MBA-subalgebra of $\mathbf{P}_m$. □

Corollary 3.3.7. *For every $1 \le n \le m \le \omega$, if $\mathbf{P}_m \in V$, then $\mathbf{P}_n \in V$.*

Lemma 3.3.8. *If $\mathbf{P}_n \in V$ for arbitrarily large $1 \le n \in \omega$ (i.e. for every $m \in \omega$ with $1 \le m$ there exists $k \in \omega$ such that $m \le k$ and $\mathbf{P}_k \in V$), then*

1. *$\mathbf{P}_n \in V$ for every $1 \le n < \omega$;*

2. *$\mathbf{P}_\omega \in V$.*

Proof. 1. Let $1 \le n < \omega$ be fixed. By assumption, there exists $k \in \omega$ such that $n \le k$ and $\mathbf{P}_k \in V$. Then $\mathbf{P}_n \in V$ by Corollary 3.3.7.

2. Assume that $\mathbf{P}_\omega \notin V$. Hence there exists an identity $t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ such that $\mathbf{P}_\omega \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ and $V \models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ (by Theorem 3.1.11). Since $\mathbf{P}_\omega \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ and $\mathcal{F}_\omega$ is a structure of Type I, we obtain by Corollary 3.2.3 that $\mathbf{A}_1 \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ where $\mathbf{A}_1 = \mathbf{P}_{\mathcal{F}'}$ for some finite bounded graph $\mathcal{F}'$ of Type I, say $Card(\mathcal{F}') = n \ge 1$. Therefore $\mathbf{A}_1$ and $\mathbf{P}_n$ are isomorphic (by Lemma 3.3.5). So $\mathbf{P}_n \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$. But $\mathbf{P}_n \in V$ (by the item (1) of this lemma) and $V \models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$. This is a contradiction. Thus $\mathbf{P}_\omega \in V$. □

Lemma 3.3.9. *For every $1 \le n \le m \le \omega$, $\mathbf{P}_n^\infty$ is isomorphic to a subalgebra of $\mathbf{P}_m^\infty$.*

Proof. Since $n \le m$, we have $n \subseteq m$. Then define a mapping $f : \{\infty\} \cup m \to \{\infty\} \cup n$ by

$$f(j) = \begin{cases} \infty, & \text{if } j = \infty \\ j, & \text{if } j \in n \\ 0, & \text{if } j \in m - n \end{cases} \tag{3.3.2}$$

for every $j \in \{\infty\} \cup m$. Obviously, f is surjective. It follows from the fact that both $\mathcal{F}_n^\infty$ and $\mathcal{F}_m^\infty$ are bounded graphs of Type II that f is a bounded morphism from $\mathcal{F}_m^\infty$ to $\mathcal{F}_n^\infty$. So $f : \mathcal{F}_m^\infty \to \mathcal{F}_n^\infty$ is a surjective bounded morphism. Therefore $h_f : \mathbf{P}_n^\infty \to \mathbf{P}_m^\infty$ is an injective MBA-homomorphism (by Corollary 3.1.27). Hence $\mathbf{P}_n^\infty$ is isomorphic to an MBA-subalgebra of $\mathbf{P}_m^\infty$. □

Corollary 3.3.10. *For every $1 \leq n \leq m \leq \omega$, if $\mathbf{P}_m^\infty \in V$, then $\mathbf{P}_n^\infty \in V$.*

Lemma 3.3.11. *If $\mathbf{P}_n^\infty \in V$ for arbitrarily large $1 \leq n \in \omega$, then*

1. *$\mathbf{P}_n^\infty \in V$ for every $1 \leq n < \omega$;*
2. *$\mathbf{P}_\omega^\infty \in V$.*

Proof. 1. Let $1 \leq n < \omega$ be fixed. By assumption, there exists $k \in \omega$ such that $n \leq k$ and $\mathbf{P}_k^\infty \in V$. Then $\mathbf{P}_n^\infty \in V$ by Corollary 3.3.10.

2. Assume that $\mathbf{P}_\omega^\infty \notin V$. Hence there exists an identity $t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ such that $\mathbf{P}_\omega^\infty \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ and $V \models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ (by Theorem 3.1.11). Since $\mathbf{P}_\omega^\infty \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ and $\mathcal{F}_\omega^\infty$ is a bounded graph of Type II, we obtain by Corollary 3.2.3 that $\mathbf{A}_1 \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$ where $\mathbf{A}_1 = \mathbf{P}_{\mathcal{F}'}$ for some finite bounded graph of Type II. Therefore $\mathbf{A}_1$ and $\mathbf{P}_n^\infty$ (for some $1 \leq n < \omega$) are isomorphic (by Lemma 3.3.5). So $\mathbf{P}_n^\infty \not\models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$. But $\mathbf{P}_n^\infty \in V$ (by the item (1) of this lemma) and $V \models t(x_0, \ldots, x_{l-1}) \approx \mathbf{1}$. This is a contradiction. Thus $\mathbf{P}_\omega^\infty \in V$. □

Lemma 3.3.12. *For every $1 \leq n \leq \omega$, $\mathbf{P}_n$ is an homomorphic image of the MBA $\mathbf{P}_n^\infty$.*

Proof. By Lemma 3.1.31 and Corollary 3.1.27. □

Corollary 3.3.13. *For every $1 \leq n \leq \omega$, if $\mathbf{P}_n^\infty \in V$, then $\mathbf{P}_n \in V$.*

We are now ready to prove the main result of the section.

Theorem 3.3.14. *Suppose V is a variety of MBA's. Then there exist $1 \leq i \leq \omega$, $1 \leq j \leq \omega$ and a subset $S \subseteq \{\mathbf{P}_i, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ such that $V = V(S)$. (Roughly speaking, every variety of MBA's is generated by at most three special members.)*

Proof. Let $K = \{\mathbf{A} \in V \mid \mathbf{A} \text{ is a finite special MBA}\}$. Recall that $V = V(K)$ by Corollary 3.2.2. Our goal is to single out in V (not necessarily in K) as few members as possible that generate the whole variety V.

Let

- $K_0 = \{\mathbf{A} \in K \mid \mathbf{A} \text{ is a (finite) special MBA of Type I}\}$,
- $K_1 = \{\mathbf{A} \in K \mid \mathbf{A} \text{ is a (finite) special MBA of Type II}\}$,
- $K_2 = \{\mathbf{A} \in K \mid \mathbf{A} \text{ is a vacuous MBA}\}$,
- $K^* = \big\{\mathbf{A} \in K \mid \mathbf{A} \in \{\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty \mid 1 \le n < \omega, 1 \le m < \omega\}\big\}$,
- $S_0 = \{\mathbf{P}_n \mid 1 \le n < \omega\}$,
- $S_1 = \{\mathbf{P}_m^\infty \mid 1 \le m < \omega\}$.

Note that $K = K_0 \cup K_1 \cup K_2$.

We are going to define three sets G, H, and I, whose union will be the desired S. Below G will consist of at most one MBA $\mathbf{P}_i$ (for some $1 \le i \le \omega$), H will consist of at most one MBA $\mathbf{P}_j^\infty$ (for some $1 \le j \le \omega$) and I will consist of at most $\mathbf{P}_0^\infty$.

Define an index i and a set G according to the following:

Case 1 If $K^* \cap S_0 = \emptyset$, then let $G = \emptyset$ and $i = 1$;

Case 2 If $1 \le Card(K^* \cap S_0) < \omega$, then let $i = \max\{n \mid \mathbf{P}_n \in K^*\}$ and $G = \{\mathbf{P}_i\}$;

Case 3 If $Card(K^* \cap S_0) = \omega$, then $\mathbf{P}_\omega \in V$ (by Lemma 3.3.8) and let $G = \{\mathbf{P}_\omega\}$ and $i = \omega$.

Similarly define an index j and a set H:

Case 1 If $K^* \cap S_1 = \emptyset$, then let $H = \emptyset$ and $j = 1$;

Case 2 If $1 \le Card(K^* \cap S_1) < \omega$, then let $j = \max\{m \mid \mathbf{P}_m^\infty \in K^*\}$ and $H = \{\mathbf{P}_j^\infty\}$;

Case 3 If $Card(K^* \cap S_1) = \omega$, then $\mathbf{P}_\omega^\infty \in V$ (by Lemma 3.3.11) and let $H = \{\mathbf{P}_\omega^\infty\}$ and $j = \omega$.

Finally, define a set I:

Case 1 If $\mathbf{P}_0^\infty \notin K^*$, then let $I = \emptyset$;

Case 2 If $\mathbf{P}_0^\infty \in K^*$, then let $I = \{\mathbf{P}_0^\infty\}$.

So let $S = G \cup H \cup I$. Thus $S \subseteq \{\mathbf{P}_i, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ and $S \subseteq V$.

Our goal is to prove that $V(S) = V$, i.e. S generates V. Let X be an infinite set of variables. By the discussion on page 50, to prove $V(S) = V$ it suffices to show that $Id_S(X) \subseteq Id_V(X)$. Moreover, since $V = V(K)$, it suffices to show that, for every MBA-term $t(x_0, \ldots, x_{n-1})$, $S \models t \approx \mathbf{1}$ implies $K \models t \approx \mathbf{1}$.

Firstly, to be proved that

$$G \models t \approx \mathbf{1} \text{ implies } K_0 \models t \approx \mathbf{1} \text{ (for every term } t). \tag{3.3.3}$$

Suppose $G \models t \approx \mathbf{1}$. By definition of G, there are three cases.

Case 1 If $K^* \cap S_0 = \emptyset$, then $K_0 = \emptyset$ (by Lemma 3.3.5). Hence $K_0 \models t \approx \mathbf{1}$.

Case 2 If $1 \leq Card(K^* \cap S_0) < \omega$, then $\mathbf{P}_i \models t \approx \mathbf{1}$ where $i = \max\{n \mid \mathbf{P}_n \in K^*\}$. Hence $\mathbf{P}_k \models t \approx \mathbf{1}$ and $\mathbf{P}_k \in V$ for every $1 \leq k \leq i$ (by Lemma 3.3.6 and Corollary 3.3.7). Therefore $K_0 \models t \approx \mathbf{1}$ (by Lemma 3.3.5 and by definition of i).

Case 3 If $Card(K^* \cap S_0) = \omega$, then $\mathbf{P}_\omega \models t \approx \mathbf{1}$. Hence $\mathbf{P}_k \models t \approx \mathbf{1}$ and $\mathbf{P}_k \in V$ for every $1 \leq k < \omega$ (by Lemma 3.3.6 and Corollary 3.3.7). Therefore $K_0 \models t \approx \mathbf{1}$ (by Lemma 3.3.5).

Secondly, to be proved that

$$H \models t \approx \mathbf{1} \text{ implies } K_1 \models t \approx \mathbf{1} \text{ (for every term } t). \tag{3.3.4}$$

Suppose $H \models t \approx \mathbf{1}$. By definition of H, there are three cases.

Case 1 If $K^* \cap S_1 = \emptyset$, then $K_1 = \emptyset$ (by Lemma 3.3.5). Hence $K_1 \models t \approx \mathbf{1}$.

Case 2 If $1 \leq Card(K^* \cap S_1) < \omega$, then $\mathbf{P}_j^\infty \models t \approx \mathbf{1}$ where $j = \max\{m \mid \mathbf{P}_m^\infty \in K^*\}$. Hence $\mathbf{P}_k^\infty \models t \approx \mathbf{1}$ for every $1 \leq k \leq j$ and $\mathbf{P}_k^\infty \in V$ (by Lemma 3.3.9 and Corollary 3.3.10). Therefore $K_1 \models t \approx \mathbf{1}$ (by Lemma 3.3.5 and by definition of j).

Case 3 If $Card(K^* \cap S_1) = \omega$, then $\mathbf{P}_\omega^\infty \models t \approx \mathbf{1}$. Hence $\mathbf{P}_k^\infty \models t \approx \mathbf{1}$ and $\mathbf{P}_k^\infty \in V$ for every $1 \leq k < \omega$ (by Lemma 3.3.9 and Corollary 3.3.10). Therefore $K_1 \models t \approx \mathbf{1}$ (by Lemma 3.3.5).

Finally, to be proved that

$$I \models t \approx \mathbf{1} \text{ implies } K_2 \models t \approx \mathbf{1} \text{ (for every term } t). \tag{3.3.5}$$

Suppose $I \models t \approx \mathbf{1}$. By definition of I, there are two cases.

Case 1 If $\mathbf{P}_0^\infty \notin K^*$, then $K_2 = \emptyset$ (by Lemma 3.3.5). Hence $K_2 \models t \approx \mathbf{1}$.

Case 2 If $\mathbf{P}_0^\infty \in K^*$, then $\mathbf{P}_0^\infty \models t \approx \mathbf{1}$. Therefore $K_2 \models t \approx \mathbf{1}$ (by Lemma 3.3.5).

It follows from (3.3.3), (3.3.4), (3.3.5) that $S \models t \approx \mathbf{1}$ implies $K \models t \approx \mathbf{1}$ (for every MBA-term t). So $V(S) = V$, i.e. V is generated by S. □

Corollary 3.3.15. *There are countably many varieties of MBA's.*

3.4 Equational characterizations of MBA-varieties

In [9] D. Monk gives explicit equational characterizations for each variety of monadic algebras. The purpose of this section is to provide an analogous result for monadic bounded algebras, namely, equationally characterize each variety of MBA's. As a consequence, we get that the equational theory of every MBA-variety is finitely based. Our algebraic expressions are obtained by modifying certain formulas from modal logic due to K. Segerberg [10].

We are going to sort all MBA-varieties via the previous section. By Theorem 3.3.14, there are eighteen types of MBA-varieties:

1. $V(\mathbf{P}_\omega, \mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty)$,
2. $V(\mathbf{P}_\omega, \mathbf{P}_\omega^\infty)$,
3. $V(\mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty)$,
4. $V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$,
5. $V(\mathbf{P}_\omega)$,
6. $V(\mathbf{P}_\omega^\infty)$,
7. $V(\mathbf{P}_0^\infty)$,
8. $V(\mathbf{P}_n, \mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty)$, $1 \leq n < \omega$,
9. $V(\mathbf{P}_n, \mathbf{P}_\omega^\infty)$, $1 \leq n < \omega$,
10. $V(\mathbf{P}_n, \mathbf{P}_0^\infty)$, $1 \leq n < \omega$,
11. $V(\mathbf{P}_n)$, $1 \leq n < \omega$,
12. $V(\mathbf{P}_\omega, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$, $1 \leq m < \omega$,
13. $V(\mathbf{P}_\omega, \mathbf{P}_m^\infty)$, $1 \leq m < \omega$,
14. $V(\mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$, $1 \leq m < \omega$,
15. $V(\mathbf{P}_m^\infty)$, $1 \leq m < \omega$,
16. $V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$, $1 \leq n < \omega$ and $1 \leq m < \omega$,
17. $V(\mathbf{P}_n, \mathbf{P}_m^\infty)$, $1 \leq n < \omega$ and $1 \leq m < \omega$,
18. $V(\emptyset)$.

Let us consider two cases both in (16) and in (17):

16a. $V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$, $1 \leq n \leq m < \omega$,

16b. $V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$, $1 \leq m < n < \omega$,

and

17a. $V(\mathbf{P}_n, \mathbf{P}_m^\infty)$, $1 \leq n \leq m < \omega$,

17b. $V(\mathbf{P}_n, \mathbf{P}_m^\infty)$, $1 \leq m < n < \omega$.

Recall that

- $\mathbf{P}_i$ is isomorphic to an MBA-subalgebra of $\mathbf{P}_j$ for $1 \leq i \leq j \leq \omega$ (see Lemma 3.3.6),
- $\mathbf{P}_i^\infty$ is isomorphic to an MBA-subalgebra of $\mathbf{P}_j^\infty$ for $1 \leq i \leq j \leq \omega$ (see Lemma 3.3.9),
- $\mathbf{P}_i$ is an homomorphic image of $\mathbf{P}_i^\infty$ for $1 \leq i \leq \omega$ (see Lemma 3.3.12).

Therefore some MBA-varieties are equal:

- $V(\mathbf{P}_\omega, \mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_n, \mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty)$ for $1 \leq n < \omega$ (see items 1, 3, 8),
- $V(\mathbf{P}_\omega, \mathbf{P}_\omega^\infty) = V(\mathbf{P}_\omega^\infty) = V(\mathbf{P}_n, \mathbf{P}_\omega^\infty)$ for $1 \leq n < \omega$ (see items 2, 6, 9),
- $V(\mathbf{P}_m^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$ for $1 \leq n \leq m < \omega$ (see items 14, 16a),
- $V(\mathbf{P}_m^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty)$ for $1 \leq n \leq m < \omega$ (see items 15, 17a).

So there are actually fourteen types of MBA-varieties:

1. $V(\mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty)$,
2. $V(\mathbf{P}_\omega^\infty)$,
3. $V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$,
4. $V(\mathbf{P}_\omega)$,
5. $V(\mathbf{P}_0^\infty)$,
6. $V(\mathbf{P}_n, \mathbf{P}_0^\infty)$, $1 \leq n < \omega$,

7. $V(\mathbf{P}_n)$, $1 \leq n < \omega$,
8. $V(\mathbf{P}_\omega, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$, $1 \leq m < \omega$,
9. $V(\mathbf{P}_\omega, \mathbf{P}^\infty_m)$, $1 \leq m < \omega$,
10. $V(\mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$, $1 \leq m < \omega$,
11. $V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$, $1 \leq m < n < \omega$,
12. $V(\mathbf{P}^\infty_m)$, $1 \leq m < \omega$,
13. $V(\mathbf{P}_n, \mathbf{P}^\infty_m)$, $1 \leq m < n < \omega$,
14. $V(\emptyset)$.

The goal is to find equations which characterize each of them.

Definition 3.4.1. *$\bar{V}$ is the variety of all MBA's.*

Definition 3.4.2. *V_0 is the variety of all one-element MBA's.*

So $V_0 = V(\emptyset)$ and $\bar{V} = V(\mathbf{P}^\infty_\omega, \mathbf{P}^\infty_0)$.
Let $\{v_0, v_1, \dots\}$ be a set of variables.

Definition 3.4.3 (cf. [10, p. 52])**.** *For $1 \leq n < \omega$, denote the MBA-term*

$$\bigwedge_{0 \leq i \leq n} \exists(v_0 \wedge \cdots \wedge v_{i-1} \wedge v_i') \tag{3.4.1}$$

by Alt_n.

Note that Alt_n is not defined for $n = \omega$.

Definition 3.4.4 (cf. [9, p. 54])**.** *A set Γ of MBA-equations characterizes a set L of MBA's relative to $\bar{V}$ (or, L is characterized by Γ relative to $\bar{V}$) iff $L = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$.*

Recall that to distinguish operations and constants in different MBA's we use superscripts:

- $\mathbf{P}_k^\infty = (\mathbf{P}_k^\infty, \cap, \cup, ', \mathbf{0}^{\mathbf{P}_k^\infty}, \mathbf{1}^{\mathbf{P}_k^\infty}, E^{\mathbf{P}_k^\infty}, \exists^{\mathbf{P}_k^\infty})$,
- $\mathbf{P}_k = (\mathbf{P}_k, \cap, \cup, ', \mathbf{0}^{\mathbf{P}_k}, \mathbf{1}^{\mathbf{P}_k}, E^{\mathbf{P}_k}, \exists^{\mathbf{P}_k})$,
- $\mathbf{P}_0^\infty = (\mathbf{P}_0^\infty, \cap, \cup, ', \mathbf{0}^{\mathbf{P}_0^\infty}, \mathbf{1}^{\mathbf{P}_0^\infty}, E^{\mathbf{P}_0^\infty}, \exists^{\mathbf{P}_0^\infty})$.

Lemma 3.4.5. *For $1 \leq n < \omega$, the following conditions are equivalent:*

1. $\mathbf{P}_k^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$;
2. $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$;
3. $k \leq n$.

In other words, $\mathbf{P}_k^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$ iff there are at most n elements in $E^{\mathbf{P}_k^\infty}$ and $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$ iff there are at most n elements in $E^{\mathbf{P}_k}$.

Proof. $(1) \Rightarrow (2)$. By Lemma 3.3.12.
$(2) \Rightarrow (3)$. Assume $k > n$. Then $k - n \geq 1$. For every $i < n$, let $p_i = \{0, \ldots, k-(i+2)\}$, and $p_n = \emptyset$. So

$$k - (i+1) \in p_0 \cap \cdots \cap p_{i-1} \cap p_i',$$

and therefore $p_0 \cap \cdots \cap p_{i-1} \cap p_i' \neq \emptyset$ (for each $i \leq n$). Hence

$$\bigcap_{i \leq n} \exists^{\mathbf{P}_k}(p_0 \cap \cdots \cap p_{i-1} \cap p_i') = \bigcap_{i \leq n}\{0, \ldots, n-1\} = \{0, \ldots, n-1\} \neq \emptyset$$

(since $E^{\mathbf{P}_k} = \mathbf{1}^{\mathbf{P}_k}$ and $\exists^{\mathbf{P}_k}$ is basic). Thus $\mathbf{P}_k \not\models \mathsf{Alt}_n \approx \mathbf{0}$.
$(3) \Rightarrow (1)$. Assume that $\mathbf{P}_k^\infty \not\models \mathsf{Alt}_n \approx \mathbf{0}$. Then

$$\bigcap_{i \leq n} \exists^{\mathbf{P}_k^\infty}(p_0 \cap \cdots \cap p_{i-1} \cap p_i') \neq \emptyset$$

for some $p_0, \ldots, p_n \in \mathbf{P}_k^\infty$ (so $p_0, \ldots, p_n \subseteq \{\infty, 0, \ldots, k-1\}$). Then each member of the intersection is nonempty. Therefore, by Definition 2.2.1 (1,6),

$$E^{\mathbf{P}_k^\infty} \cap p_0' \neq \emptyset,$$

$$E^{\mathbf{P}_k^\infty} \cap p_0 \cap p_1' \neq \emptyset,$$

$$E^{\mathbf{P}_k^\infty} \cap p_0 \cap p_1 \cap p_2' \neq \emptyset,$$

$$\dots$$

$$E^{\mathbf{P}_k^\infty} \cap p_0 \cap \dots \cap p_{n-2} \cap p'_{n-1} \neq \emptyset,$$

$$E^{\mathbf{P}_k^\infty} \cap p_0 \cap \dots \cap p_{n-1} \cap p'_n \neq \emptyset.$$

So we have that
1) $E^{\mathbf{P}_k^\infty}$ consists of $k > 0$ elements (by definition of $\mathbf{P}_k^\infty$),
2) $E^{\mathbf{P}_k^\infty} \cap p_0$ consists of at most $k - 1 > 0$ elements,
3) $E^{\mathbf{P}_k^\infty} \cap p_0 \cap p_1$ consists of at most $k - 2 > 0$ elements,
...
n) $E^{\mathbf{P}_k^\infty} \cap p_0 \cap \dots \cap p_{n-2}$ consists of at most $k - (n-1) > 0$ elements,
n+1) $E^{\mathbf{P}_k^\infty} \cap p_0 \cap \dots \cap p_{n-1}$ consists of at most $k - n > 0$ elements.
Thus $k > n$. □

Lemma 3.4.6. *For every* $1 \leq n < \omega$, $\mathbf{P}_0^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$.

Proof. Since $\exists^{\mathbf{P}_0^\infty} p = \emptyset$ for every $p \in \mathbf{P}_0^\infty$, we obtain the result. □

Lemma 3.4.7. *For* $1 \leq m < n \leq \omega$, $V(\mathbf{P}_m) \subset V(\mathbf{P}_n)$.

Proof. Suppose $1 \leq m < n \leq \omega$. Then $\mathbf{P}_m$ is isomorphic to an MBA-subalgebra of $\mathbf{P}_n$ (by Lemma 3.3.6). Therefore $V(\mathbf{P}_m) \subseteq V(\mathbf{P}_n)$. By Lemma 3.4.5, $\mathbf{P}_m \models \mathsf{Alt}_m \approx \mathbf{0}$. But $\mathbf{P}_n \not\models \mathsf{Alt}_m \approx \mathbf{0}$ (otherwise $n \leq m$). Thus $V(\mathbf{P}_m) \neq V(\mathbf{P}_n)$. So $V(\mathbf{P}_m) \subset V(\mathbf{P}_n)$. □

Lemma 3.4.8. *For* $1 \leq m < n \leq \omega$, $V(\mathbf{P}_m^\infty) \subset V(\mathbf{P}_n^\infty)$.

Proof. Suppose $1 \leq m < n \leq \omega$. Then $\mathbf{P}_m^\infty$ is isomorphic to an MBA-subalgebra of $\mathbf{P}_n^\infty$ (by Lemma 3.3.9). Therefore $V(\mathbf{P}_m^\infty) \subseteq V(\mathbf{P}_n^\infty)$. By Lemma 3.4.5, $\mathbf{P}_m^\infty \models \mathsf{Alt}_m \approx \mathbf{0}$. But $\mathbf{P}_n^\infty \not\models \mathsf{Alt}_m \approx \mathbf{0}$ (otherwise $n \leq m$). Thus $V(\mathbf{P}_m^\infty) \neq V(\mathbf{P}_n^\infty)$. So $V(\mathbf{P}_m^\infty) \subset V(\mathbf{P}_n^\infty)$. □

Lemma 3.4.9. *For every variety V of MBA's, if*

- $\mathbf{P}_0^\infty \in V$ *or*

- $\mathbf{P}_n \in V$ *(for some* $1 \leq n \leq \omega$*) or*
- $\mathbf{P}_m^\infty \in V$ *(for some* $1 \leq m \leq \omega$*),*

then $V \neq V_0$.

Proof. Obvious. □

Lemma 3.4.10. *For* $1 \leq k \leq \omega$ *and* $1 \leq n < \omega$,

$$\mathbf{P}_k^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1} \textit{ iff } \mathbf{P}_k^\infty \models \mathsf{Alt}_n \approx \mathbf{0}.$$

Proof. The $\Leftarrow$ part is obvious. For $\Rightarrow$, assume $\mathbf{P}_k^\infty \not\models \mathsf{Alt}_n \approx \mathbf{0}$. Then

$$\textstyle\bigcap_{i \leq n} \exists^{\mathbf{P}_k^\infty}(p_0 \cap \cdots \cap p_{i-1} \cap p_i') \neq \emptyset$$

for some $p_0, \ldots, p_n \subseteq \{\infty, 0, \ldots, n-1\}$. Therefore

$$\textstyle\bigcap_{i \leq n} \exists^{\mathbf{P}_k^\infty}(p_0 \cap \cdots \cap p_{i-1} \cap p_i') = \{\infty, 0, \ldots, n-1\}$$

(since $\mathbf{P}_k^\infty$ is basic). Hence

$$\left(\textstyle\bigcap_{i \leq n} \exists^{\mathbf{P}_k^\infty}(p_0 \cap \cdots \cap p_{i-1} \cap p_i')\right)' = \emptyset.$$

So

$$\mathbf{P}_k^\infty \not\models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$$

(since $E^{\mathbf{P}_k^\infty} = \{0, \ldots, k-1\} \neq \{\infty, 0, \ldots, k-1\}$). □

Corollary 3.4.11. *For* $1 \leq k \leq \omega$ *and* $1 \leq n < \omega$, $\mathbf{P}_k^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$ *iff* $k \leq n$.

The next technical lemma about certain inequalitites among MBA-varieties will be applied many times in Facts 3.4.14-3.4.27.

Lemma 3.4.12. *Let* S *be a set of MBA's.*

1. *For* $1 \leq n \leq \omega$, $V(\mathbf{P}_n) \neq V(S)$ *where either* $\mathbf{P}_m^\infty \in S$ *(for some* $1 \leq m \leq \omega$*) or* $\mathbf{P}_0^\infty \in S$.

2. *$V(\mathbf{P}_0^\infty) \neq V(S)$ where either $\mathbf{P}_n \in S$ (for some $1 \leq n \leq \omega$) or $\mathbf{P}_m^\infty \in S$ (for some $1 \leq m \leq \omega$).*

3. *For $1 \leq j \leq \omega$, $V(\mathbf{P}_j^\infty) \neq V(S)$ where $\mathbf{P}_0^\infty \in S$.*

4. *For $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$, $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \neq V(S)$ where $\mathbf{P}_0^\infty \in S$.*

5. *For $1 \leq k \leq \omega$, $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \neq V(S)$ where $\mathbf{P}_m^\infty \in S$ (for some $1 \leq m \leq \omega$).*

6. *For $1 \leq k \leq \omega$ and $1 \leq j < m \leq \omega$, $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \neq V(\mathbf{P}_m^\infty)$.*

7. *For $1 \leq j \leq m < n \leq \omega$, $V(\mathbf{P}_j^\infty) \neq V(\mathbf{P}_n, \mathbf{P}_m^\infty)$.*

8. *For $1 \leq j \leq m < n \leq \omega$, $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \neq V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$.*

Proof. 1. Assume $V(\mathbf{P}_n) = V(S)$ where n and S satisfy the given conditions. Since $\mathbf{P}_n \models E \approx \mathbf{1}$, we have $V(\mathbf{P}_n) \models E \approx \mathbf{1}$. Then $V(S) \models E \approx \mathbf{1}$. Hence either $\mathbf{P}_m^\infty \models E \approx \mathbf{1}$ ($1 \leq m \leq \omega$) or $\mathbf{P}_0^\infty \models E \approx \mathbf{1}$. But $E^{\mathbf{P}_m^\infty} \neq \mathbf{1}^{\mathbf{P}_m^\infty}$ and $E^{\mathbf{P}_0^\infty} \neq \mathbf{1}^{\mathbf{P}_0^\infty}$.

2. Assume $V(\mathbf{P}_0^\infty) = V(S)$ where S satisfies the given condition. Since $\mathbf{P}_0^\infty \models E \approx \mathbf{0}$, we have $V(\mathbf{P}_0^\infty) \models E \approx \mathbf{0}$. Then $V(S) \models E \approx \mathbf{0}$. Hence either $\mathbf{P}_n \models E \approx \mathbf{0}$ ($1 \leq n \leq \omega$) or $\mathbf{P}_m^\infty \models E \approx \mathbf{0}$ ($1 \leq m \leq \omega$). But $E^{\mathbf{P}_n} \neq \mathbf{0}^{\mathbf{P}_n}$ and $E^{\mathbf{P}_m^\infty} \neq \mathbf{0}^{\mathbf{P}_m^\infty}$.

3. Assume $V(\mathbf{P}_j^\infty) = V(S)$ where j and S satisfy the given conditions. Since $\mathbf{P}_j^\infty \models \exists E \approx \mathbf{1}$, we have $V(\mathbf{P}_j^\infty) \models \exists E \approx \mathbf{1}$. Then $V(S) \models \exists E \approx \mathbf{1}$. Hence $\mathbf{P}_0^\infty \models \exists E \approx \mathbf{1}$. But $\exists^{\mathbf{P}_0^\infty} E^{\mathbf{P}_0^\infty} \neq \mathbf{1}^{\mathbf{P}_0^\infty}$.

4. Assume $V(\mathbf{P}_k, \mathbf{P}_j^\infty) = V(S)$ where k, j, S satisfy the given conditions. Since $\mathbf{P}_k, \mathbf{P}_j^\infty \models \exists E \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \models \exists E \approx \mathbf{1}$. Then $V(S) \models \exists E \approx \mathbf{1}$. Hence $\mathbf{P}_0^\infty \models \exists E \approx \mathbf{1}$. But $\exists^{\mathbf{P}_0^\infty} E^{\mathbf{P}_0^\infty} \neq \mathbf{1}^{\mathbf{P}_0^\infty}$.

5. Assume $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \neq V(S)$ where k and S satisfy the given conditions. Since $\mathbf{P}_k, \mathbf{P}_0^\infty \models E \vee (\exists E)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \models E \vee (\exists E)' \approx \mathbf{1}$. Then $V(S) \models E \vee (\exists E)' \approx \mathbf{1}$. Hence $\mathbf{P}_m^\infty \models E \vee (\exists E)' \approx \mathbf{1}$ ($1 \leq m \leq \omega$). But $\mathbf{P}_m^\infty \not\models E \vee (\exists E)' \approx \mathbf{1}$.

6. Assume $V(\mathbf{P}_k, \mathbf{P}_j^\infty) = V(\mathbf{P}_m^\infty)$ where k, j, m satisfy the given conditions. Since $\mathbf{P}_k, \mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$.

Then $V(\mathbf{P}_m^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Hence $\mathbf{P}_m^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Therefore $m \leq j$ (by Corollary 3.4.11). But $j < m$.

7. Assume $V(\mathbf{P}_j^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty)$ where j, n, m satisfy the given conditions. Since $\mathbf{P}_j^\infty \models \mathsf{Alt}_j \approx \mathbf{0}$, we have $V(\mathbf{P}_j^\infty) \models \mathsf{Alt}_j \approx \mathbf{0}$. Then $V(\mathbf{P}_n, \mathbf{P}_m^\infty) \models \mathsf{Alt}_j \approx \mathbf{0}$. Hence $\mathbf{P}_n \models \mathsf{Alt}_j \approx \mathbf{0}$. Therefore $n \leq j$ (by Lemma 3.4.5). So $n < n$.

8. Assume $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$ where j, n, m satisfy the given conditions. Since $\mathbf{P}_j^\infty, \mathbf{P}_0^\infty \models \mathsf{Alt}_j \approx \mathbf{0}$, we have $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \models \mathsf{Alt}_j \approx \mathbf{0}$. Then $V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty) \models \mathsf{Alt}_j \approx \mathbf{0}$. Hence $\mathbf{P}_n \models \mathsf{Alt}_j \approx \mathbf{0}$. Therefore $n \leq j$ (by Lemma 3.4.5). So $n < n$. □

The main result of the section is the following theorem, whose proof is given in a series of fourteen facts.

Theorem 3.4.13. *$V(S)$ is characterized by a finite set Γ of MBA-equations relative to $\bar{V}$, where S and Γ are in the table below.*

Proof. See Facts 3.4.14-3.4.27. □

Fact	S	Γ
3.4.14	$\{\mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty\}$	$\emptyset$
3.4.15	$\{\mathbf{P}_\omega^\infty\}$	$\{\exists E \approx \mathbf{1}\}$
3.4.16	$\{\mathbf{P}_\omega, \mathbf{P}_0^\infty\}$	$\{E \vee (\exists E)' \approx \mathbf{1}\}$
3.4.17	$\{\mathbf{P}_\omega\}$	$\{E \approx \mathbf{1}\}$
3.4.18	$\{\mathbf{P}_0^\infty\}$	$\{E \approx \mathbf{0}\}$
3.4.19	$\{\mathbf{P}_n, \mathbf{P}_0^\infty\}, 1 \leq n < \omega$	$\{E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$
3.4.20	$\{\mathbf{P}_n\}, 1 \leq n < \omega$	$\{E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$
3.4.21	$\{\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty\}, 1 \leq n < \omega$	$\{E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$
3.4.22	$\{\mathbf{P}_\omega, \mathbf{P}_n^\infty\}, 1 \leq n < \omega$	$\{\exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$
3.4.23	$\{\mathbf{P}_n^\infty, \mathbf{P}_0^\infty\}, 1 \leq n < \omega$	$\{\mathsf{Alt}_n \approx \mathbf{0}\}$
3.4.24	$\{\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty\}, 1 \leq m < n < \omega$	$\{\mathsf{Alt}_n \approx \mathbf{0}, E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}\}$
3.4.25	$\{\mathbf{P}_n^\infty\}, 1 \leq n < \omega$	$\{\exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$
3.4.26	$\{\mathbf{P}_n, \mathbf{P}_m^\infty\}, 1 \leq m < n < \omega$	$\{\exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}, E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}\}$
3.4.27	$\emptyset$	$\{v_0 \approx v_1\}$

In Facts 3.4.14-3.4.27, we characterize each of the fourteen types of MBA-varieties. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Sigma\}$, where Σ is a set of MBA-equations, is an equational class of MBA's, $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Sigma\}$ is an MBA-variety (by Theorem 3.1.11). Therefore, by Theorem 3.3.14, there exist $1 \leq i \leq \omega$, $1 \leq j \leq \omega$ and a subset $S \subseteq \{\mathbf{P}_i, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ such that $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Sigma\}$. We stipulate that in the proofs Facts 3.4.14-3.4.27 all this will be shortened to the phrase "Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Sigma\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Sigma\}$ where S consists of at most three special MBA's".

Fact 3.4.14. *$V(\mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty)(= \bar{V})$ is characterized by $\emptyset$ relative to $\bar{V}$.*

Proof. Since $\bar{V} = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \emptyset\}$, $\bar{V}$ is characterized by $\emptyset$ relative to $\bar{V}$. So $V(\mathbf{P}_\omega^\infty, \mathbf{P}_0^\infty)$ is characterized by $\emptyset$ relative to $\bar{V}$. □

Fact 3.4.15. *$V(\mathbf{P}_\omega^\infty)$ $(= V(\mathbf{P}_n, \mathbf{P}_\omega^\infty)$ for every $1 \leq n \leq \omega)$ is characterized by $\{\exists E \approx \mathbf{1}\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\mathbf{P}_\omega^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}\}$. Since $\mathbf{P}_\omega^\infty \models \exists E \approx \mathbf{1}$, we have $V(\mathbf{P}_\omega^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since $\mathbf{P}_0^\infty \not\models \exists E \approx \mathbf{1}$, we have $\mathbf{P}_0^\infty \notin S$.

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_\omega) \subseteq V(\mathbf{P}_\omega^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_\omega^\infty)$. But $V(\mathbf{P}_k) \neq V(\mathbf{P}_\omega^\infty)$ (by Lemma 3.4.12 (1)).

Assume $S = \{\mathbf{P}_m^\infty\}$ for some $1 \leq m < \omega$. Then $V(\mathbf{P}_m^\infty) \subset V(\mathbf{P}_\omega^\infty)$ [by Lemma 3.4.8] $\subseteq V(S) = V(\mathbf{P}_m^\infty)$. Hence $V(\mathbf{P}_m^\infty) \subset V(\mathbf{P}_m^\infty)$.

Assume $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty\}$ for some $1 \leq k \leq \omega$ and $1 \leq j < \omega$. Then $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_\omega^\infty) = V(\mathbf{P}_\omega^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_j^\infty) = V(\mathbf{P}_\omega^\infty)$. But $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \neq V(\mathbf{P}_\omega^\infty)$ (by Lemma 3.4.12(6)).

So it remains that either $S = \{\mathbf{P}_n, \mathbf{P}_\omega^\infty\}$ (for some $1 \leq n \leq \omega$) or $S = \{\mathbf{P}_\omega^\infty\}$. Since $V(\mathbf{P}_n, \mathbf{P}_\omega^\infty) = V(\mathbf{P}_\omega^\infty)$ (for all $1 \leq n \leq \omega$), we have proved that $V(\mathbf{P}_\omega^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}\}$ ($= V(\mathbf{P}_n, \mathbf{P}_\omega^\infty)$ for every $1 \leq n \leq \omega$). □

Fact 3.4.16. *$V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$ is characterized by $\{E \vee (\exists E)' \approx \mathbf{1}\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\mathbf{P}_\omega, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}\}$. Since $\mathbf{P}_\omega, \mathbf{P}_0^\infty \models E \vee (\exists E)' \approx \mathbf{1}$, we have $V(\mathbf{P}_\omega, \mathbf{P}_0^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since $\mathbf{P}_n^\infty \not\models E \vee (\exists E)' \approx \mathbf{1}$, we have $\mathbf{P}_n^\infty \notin S$ for any $1 \leq n \leq \omega$.

Assume $S = \{\mathbf{P}_0^\infty\}$. Then $V(\mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_0^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(2)).

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_\omega) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k) \neq V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(1)).

So it remains that $S = \{\mathbf{P}_k, \mathbf{P}_0^\infty\}$ for some $1 \leq k \leq \omega$. We are going to prove that $k = \omega$. Assume $k < \omega$. Then $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_0^\infty)$. Since $\mathbf{P}_k, \mathbf{P}_0^\infty \models \mathsf{Alt}_k \approx \mathbf{0}$, we have $\mathbf{P}_\omega \models \mathsf{Alt}_k \approx \mathbf{0}$. Hence $\omega \leq k$ (by Lemma 3.4.5). But $k < \omega$ by our assumption.

So $S = \{\mathbf{P}_\omega, \mathbf{P}_0^\infty\}$. Thus $V(\mathbf{P}_\omega, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}\}$. □

Fact 3.4.17. *$V(\mathbf{P}_\omega)$ is characterized by $\{E \approx \mathbf{1}\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\mathbf{P}_\omega) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}\}$. Since $\mathbf{P}_\omega \models E \approx \mathbf{1}$, we have $V(\mathbf{P}_\omega) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since neither $\mathbf{P}_m^\infty$ (for any $1 \leq m \leq \omega$) nor $\mathbf{P}_0^\infty$ satisfies $E \approx \mathbf{1}$, we obtain that $S = \{\mathbf{P}_n\}$ for some $1 \leq n \leq \omega$. Assume $1 \leq n < \omega$. Then $V(\mathbf{P}_n) \subset$

$V(\mathbf{P}_\omega)$ [by Lemma 3.4.7] $\subseteq V(S) = V(\mathbf{P}_n)$ and so $V(\mathbf{P}_n) \subset V(\mathbf{P}_n)$. Therefore $n = \omega$. Hence $S = \{\mathbf{P}_\omega\}$. Thus $V(\mathbf{P}_\omega) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}\}$. $\square$

Fact 3.4.18. *$V(\mathbf{P}_0^\infty)$ is characterized by $\{E \approx \mathbf{0}\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{0}\}$. Since $\mathbf{P}_0^\infty \models E \approx \mathbf{0}$, we have $V(\mathbf{P}_0^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{0}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{0}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{0}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since neither $\mathbf{P}_n$ (for any $1 \leq n \leq \omega$) nor $\mathbf{P}_m^\infty$ (for any $1 \leq m \leq \omega$) satisfies $E \approx \mathbf{0}$, we obtain that $S = \{\mathbf{P}_0^\infty\}$. Thus $V(\mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{0}\}$. $\square$

Fact 3.4.19. *For $1 \leq n < \omega$, $V(\mathbf{P}_n, \mathbf{P}_0^\infty)$ is characterized by $\{E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\mathbf{P}_n, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$. Since $\mathbf{P}_n, \mathbf{P}_0^\infty \models E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}$, we have $V(\mathbf{P}_n, \mathbf{P}_0^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since $\mathbf{P}_j^\infty \not\models E \vee (\exists E)' \approx \mathbf{1}$, we have $\mathbf{P}_j^\infty \notin S$ for any $1 \leq j \leq \omega$.

Assume $S = \{\mathbf{P}_0^\infty\}$. Then $V(\mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_0^\infty) = V(\mathbf{P}_n, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_0^\infty) \neq V(\mathbf{P}_n, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(2)).

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \leq n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_n)$ [since $k \leq n$] $\subseteq (\mathbf{P}_n, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_n, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k) \neq V(\mathbf{P}_n, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(1)).

So it remains that $S = \{\mathbf{P}_k, \mathbf{P}_0^\infty\}$ for some $1 \leq k \leq \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \leq n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n, \mathbf{P}_0^\infty)$ [since $k \leq n$] $\subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_0^\infty) = V(\mathbf{P}_n, \mathbf{P}_0^\infty)$. Since $\mathbf{P}_k, \mathbf{P}_0^\infty \models \mathsf{Alt}_k \approx \mathbf{0}$, we have $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \models \mathsf{Alt}_k \approx \mathbf{0}$. Then $V(\mathbf{P}_n, \mathbf{P}_0^\infty) \models$

$\mathsf{Alt}_k \approx \mathbf{0}$ and so $\mathbf{P}_n \models \mathsf{Alt}_k \approx \mathbf{0}$. Therefore $n \leq k$ (by Lemma 3.4.5). Hence $k = n$. So $S = \{\mathbf{P}_n, \mathbf{P}_0^\infty\}$. Thus $V(\mathbf{P}_n, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$. □

Fact 3.4.20. *For* $1 \leq n < \omega$, $V(\mathbf{P}_n)$ *is characterized by* $\{E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ *relative to* $\bar{V}$.

Proof. To be proved that $V(\mathbf{P}_n) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$. Since $\mathbf{P}_n \models E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}$, we have $V(\mathbf{P}_n) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since neither $\mathbf{P}_m^\infty$ (for any $1 \leq m \leq \omega$) nor $\mathbf{P}_0^\infty$ satisfies $E \approx \mathbf{1}$, we obtain that $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Then $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$. Hence $k \leq n$ (by Lemma 3.4.5). If $k < n$, then $V(\mathbf{P}_k) \subset V(\mathbf{P}_n)$ [by Lemma 3.4.7] $\subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) \subset V(\mathbf{P}_k)$. Therefore $k = n$. So $S = \{\mathbf{P}_n\}$. Thus $V(\mathbf{P}_n) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$. □

Fact 3.4.21. *For* $1 \leq n < \omega$, $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ *is characterized by* $\{E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$ *relative to* $\bar{V}$.

Proof. To be proved that $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$. Since $\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$, we have $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Assume $S = \{\mathbf{P}_0^\infty\}$. Then $V(\mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_0^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(2)).

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_\omega) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(1)).

Assume $S = \{\mathbf{P}_j^\infty\}$ for some $1 \leq j \leq \omega$. Since $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$, we have $j \leq n$ (by Corollary 3.4.11). Then $V(\mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_n^\infty)$ [since $j \leq n$]

$\subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_j^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_j^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(3)).

Assume $S = \{\mathbf{P}_k, \mathbf{P}_0^\infty\}$ for some $1 \leq k \leq \omega$. Then $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(5)).

Assume $S = \{\mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ for some $1 \leq j \leq \omega$. Since $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$, we have $j \leq n$ (by Corollary 3.4.11). Then $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ [since $j \leq n$] $\subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(8)).

Assume $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty\}$ for some $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$. Since $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$, we have $j \leq n$ (by Corollary 3.4.11). Then $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_k, \mathbf{P}_n^\infty)$ [since $j \leq n$] $\subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_j^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(4)).

So it remains that $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ for some $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$. Since $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$, we have $j \leq n$ (by Corollary 3.4.11). Then $V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ [since $k \leq \omega$ and $j \leq n$] $\subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$.

Firstly, we are going to prove that $j = n$. Since $\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Then $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$ and so $\mathbf{P}_n^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Therefore $n \leq j$ (by Corollary 3.4.11). Hence $j = n$. Thus $S = \{\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty\}$ and $V(\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$.

Secondly, we are going to prove that $k > n$. Assume $k \leq n$. Then $V(\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. Hence $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(8)).

Thirdly, we are going to prove that $k = \omega$. Assume $k < \omega$. Since $k > n$, we have $\mathbf{P}_n^\infty \models \mathsf{Alt}_k \approx \mathbf{0}$ (by Lemma 3.4.5). Hence $\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty \models \mathsf{Alt}_k \approx \mathbf{0}$ and $V(\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \models \mathsf{Alt}_k \approx \mathbf{0}$. Then $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \models \mathsf{Alt}_k \approx \mathbf{0}$ and so $\mathbf{P}_\omega \models \mathsf{Alt}_k \approx \mathbf{0}$. Therefore $\omega \leq k$ (by Lemma 3.4.5). But $k < \omega$ by our assumption.

So $S = \{\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}^\infty\}$. Thus $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$. □

Fact 3.4.22. *For* $1 \leq n < \omega$, $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$ *is characterized by* $\{\exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$.

Proof. To be proved that $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$. Since $\mathbf{P}_\omega, \mathbf{P}_n^\infty \models \exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$, we have $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since $\mathbf{P}_0^\infty \not\models \exists E \approx \mathbf{1}$, we have $\mathbf{P}_0^\infty \notin S$.

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_\omega) \subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$. But $V(\mathbf{P}_k) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$ (by Lemma 3.4.12(1)).

Assume $S = \{\mathbf{P}_j^\infty\}$ for some $1 \leq j \leq \omega$. Then $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$. Hence $j \leq n$ (by Corollary 3.4.11). Therefore $V(\mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_n^\infty)$ [since $j \leq n$] $\subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) \subseteq V(S) = V(\mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_j^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$. But $V(\mathbf{P}_j^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$ (by Lemma 3.4.12(7)).

So it remains that $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty\}$ for some $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$. Then $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$. Hence $j \leq n$ (by Corollary 3.4.11).

We are going to prove that $j = n$. Since $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_k, \mathbf{P}_n^\infty)$ [since $j \leq n$] $\subseteq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty)$, we get $V(\mathbf{P}_k, \mathbf{P}_j^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$. Since $\mathbf{P}_k, \mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Then $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$ and so $\mathbf{P}_n^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Therefore $n \leq j$ (by Corollary 3.4.11). Hence $j = n$.

So $S = \{\mathbf{P}_k, \mathbf{P}_n^\infty\}$. Thus $V(\mathbf{P}_k, \mathbf{P}_n^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$.

If $k \leq n$, then $V(\mathbf{P}_k, \mathbf{P}_n^\infty) = V(\mathbf{P}_n^\infty)$ and so $V(\mathbf{P}_n^\infty) = V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$. But $V(\mathbf{P}_n^\infty) \neq V(\mathbf{P}_\omega, \mathbf{P}_n^\infty)$ (by Lemma 3.4.12(7)). Therefore $k > n$.

We are going to prove that $k = \omega$. Assume $1 \leq k < \omega$ (so $n < k < \omega$). Then $\mathbf{P}_k \models \mathsf{Alt}_k \approx \mathbf{0}$ and $\mathbf{P}_n^\infty \models \mathsf{Alt}_k \approx \mathbf{0}$ (by Lemma 3.4.5). Hence

$V(\mathbf{P}_k, \mathbf{P}_n^\infty) \models \mathsf{Alt}_k \approx \mathbf{0}$. Then $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) \models \mathsf{Alt}_k \approx \mathbf{0}$ and so $\mathbf{P}_\omega \models \mathsf{Alt}_k \approx \mathbf{0}$. Therefore $\omega \leq k$ (by Lemma 3.4.5). But $k < \omega$ by assumption. Hence $k = \omega$.

So $S = \{\mathbf{P}_\omega, \mathbf{P}_n^\infty\}$. Thus $V(\mathbf{P}_\omega, \mathbf{P}_n^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$. □

Fact 3.4.23. *For* $1 \leq n < \omega$, $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)(= V(\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty))$ *for any* $1 \leq k \leq n$) *is characterized by* $\{\mathsf{Alt}_n \approx \mathbf{0}\}$ *relative to* $\bar{V}$.

Proof. To be proved that $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \mathsf{Alt}_n \approx \mathbf{0}\}$. Since $\mathbf{P}_n^\infty, \mathbf{P}_0^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \mathsf{Alt}_n \approx \mathbf{0}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \mathsf{Alt}_n \approx \mathbf{0}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \mathsf{Alt}_n \approx \mathbf{0}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Assume $S = \{\mathbf{P}_0^\infty\}$. Then $V(\mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_0^\infty) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_0^\infty) \neq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(2)).

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \leq n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_n)$ [since $k \leq n$] $\subseteq V(\mathbf{P}_n^\infty) \subseteq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k) \neq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(1)).

Assume $S = \{\mathbf{P}_j^\infty\}$ for some $1 \leq j \leq \omega$. Since $\mathbf{P}_j^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $j \leq n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_n^\infty)$ [since $j \leq n$] $\subseteq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_j^\infty) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_j^\infty) \neq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(3)).

Assume $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty\}$ for some $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$ and $\mathbf{P}_j^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \leq n$ and $j \leq n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_n, \mathbf{P}_n^\infty)$ [since $k, j \leq n$] $\subseteq V(\mathbf{P}_n^\infty) \subseteq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_j^\infty) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \neq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(4)).

Assume $S = \{\mathbf{P}_k, \mathbf{P}_0^\infty\}$ for some $1 \leq k \leq \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \leq n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n, \mathbf{P}_0^\infty)$ [since $k \leq n$] $\subseteq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_0^\infty) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_k, \mathbf{P}_0^\infty) \neq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(5)).

So it remains that either $S = \{\mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ for some $1 \le j \le \omega$ or $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ for some $1 \le k \le \omega$ and $1 \le j \le \omega$.

- Suppose $S = \{\mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ for some $1 \le j \le \omega$. Since $\mathbf{P}_j^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $j \le n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ [since $j \le n$] $\subseteq V(S) = V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. Since $\mathbf{P}_j^\infty, \mathbf{P}_0^\infty \models \mathsf{Alt}_j \approx \mathbf{0}$, we have $V(\mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \models \mathsf{Alt}_j \approx \mathbf{0}$. Then $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \models \mathsf{Alt}_j \approx \mathbf{0}$ and so $\mathbf{P}_n^\infty \models \mathsf{Alt}_j \approx \mathbf{0}$. Therefore $n \le j$ (by Lemma 3.4.5). Hence $j = n$. Thus $S = \{\mathbf{P}_n^\infty, \mathbf{P}_0^\infty\}$ (in this case).

- Now suppose $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty\}$ for some $1 \le k \le \omega$ and $1 \le j \le \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$ and $\mathbf{P}_j^\infty \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \le n$ and $j \le n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ [since $k, j \le n$] $= V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$. Since $\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_j^\infty, \mathbf{P}_0^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Then $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$ and so $\mathbf{P}_n^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Therefore $n \le j$ (by Corollary 3.4.11). Hence $j = n$. Thus $S = \{\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty\}$ and $k \le n$ (in this case).

So either $S = \{\mathbf{P}_n^\infty, \mathbf{P}_0^\infty\}$ or $S = \{\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty\}$ for some $1 \le k \le n$. Since $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = V(\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ for all $1 \le k \le n$, we have proved that $V(\mathbf{P}_n^\infty, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \mathsf{Alt}_n \approx \mathbf{0}\} (= V(\mathbf{P}_k, \mathbf{P}_n^\infty, \mathbf{P}_0^\infty)$ for any $1 \le k \le n)$. □

Fact 3.4.24. *For* $1 \le m < n < \omega$, $V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$ *is characterized by* $\Gamma = \{\mathsf{Alt}_n \approx \mathbf{0}, E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}\}$ *relative to* $\bar{V}$.

Proof. To be proved that $V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$. Since $\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty \models \Gamma$, we have $V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Assume $S = \{\mathbf{P}_0^\infty\}$. Then $V(\mathbf{P}_0^\infty) \subseteq V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty) \subseteq V(S) = V(\mathbf{P}_0^\infty)$ and so $V(\mathbf{P}_0^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$. But $V(\mathbf{P}_0^\infty) \neq V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}_0^\infty)$ (by Lemma 3.4.12(2)).

Assume $S = \{\mathbf{P}_k\}$ for some $1 \le k \le \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \le n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_n)$ [since $k \le n$] $\subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$. But $V(\mathbf{P}_k) \ne V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$ (by Lemma 3.4.12(1)).

Assume $S = \{\mathbf{P}^\infty_j\}$ for some $1 \le j \le \omega$. Since $\mathbf{P}^\infty_j \models E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}$, we have $j \le m$ (by Corollary 3.4.11). Then $V(\mathbf{P}^\infty_j) \subseteq V(\mathbf{P}^\infty_m)$ [since $j \le m$] $\subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty) \subseteq V(S) = V(\mathbf{P}^\infty_j)$ and so $V(\mathbf{P}^\infty_j) = V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$. But $V(\mathbf{P}^\infty_j) \ne V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$ (by Lemma 3.4.12(3)).

Assume $S = \{\mathbf{P}_k, \mathbf{P}^\infty_0\}$ for some $1 \le k \le \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$, we have $k \le n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k, \mathbf{P}^\infty_0) \subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_0)$ [since $k \le n$] $\subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}^\infty_0)$ and so $V(\mathbf{P}_k, \mathbf{P}^\infty_0) = V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$. But $V(\mathbf{P}_k, \mathbf{P}^\infty_0) \ne V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$ (by Lemma 3.4.12(5)).

Assume $S = \{\mathbf{P}_k, \mathbf{P}^\infty_j\}$ for some $1 \le k \le \omega$ and $1 \le j \le \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$ and $\mathbf{P}^\infty_j \models E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}$, we have $k \le n$ (by Lemma 3.4.5) and $j \le m$ (by Corollary 3.4.11). Then $V(\mathbf{P}_k, \mathbf{P}^\infty_j) \subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_m)$ [since $k \le n$ and $j \le m$] $\subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}^\infty_j)$ and so $V(\mathbf{P}_k, \mathbf{P}^\infty_j) = V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$. But $V(\mathbf{P}_k, \mathbf{P}^\infty_j) \ne V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$ (by Lemma 3.4.12(4)).

Assume $S = \{\mathbf{P}^\infty_j, \mathbf{P}^\infty_0\}$ for some $1 \le j \le \omega$. Since $\mathbf{P}^\infty_j \models E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}$, we have $j \le m$ (by Corollary 3.4.11). Then $V(\mathbf{P}^\infty_j, \mathbf{P}^\infty_0) \subseteq V(\mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$ [since $j \le m$] $\subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) \subseteq V(S) = V(\mathbf{P}^\infty_j, \mathbf{P}^\infty_0)$ and so $V(\mathbf{P}^\infty_j, \mathbf{P}^\infty_0) = V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$. But $V(\mathbf{P}^\infty_j, \mathbf{P}^\infty_0) \ne V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$ (by Lemma 3.4.12(8)).

So it remains that $S = \{\mathbf{P}_k, \mathbf{P}^\infty_j, \mathbf{P}^\infty_0\}$ for some $1 \le k \le \omega$ and $1 \le j \le \omega$. Since $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$ and $\mathbf{P}^\infty_j \models E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}$, we have $k \le n$ (by Lemma 3.4.5) and $j \le m$ (by Corollary 3.4.11). Then $V(\mathbf{P}_k, \mathbf{P}^\infty_j, \mathbf{P}^\infty_0) \subseteq V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty)$ [since $k \le n$ and $j \le m$] $\subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}^\infty_j, \mathbf{P}^\infty_0)$ and so $V(\mathbf{P}_k, \mathbf{P}^\infty_j, \mathbf{P}^\infty_0) = V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$.

Firstly, we are going to prove that $j = m$. Since $\mathbf{P}_k, \mathbf{P}^\infty_j, \mathbf{P}^\infty_0 \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}^\infty_j, \mathbf{P}^\infty_0) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Then $V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$ and so $\mathbf{P}^\infty_m \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Therefore $m \le j$ (by Corollary 3.4.11). Hence $j = m$. Thus $S = \{\mathbf{P}_k, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0\}$ and $V(\mathbf{P}_k, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) =$

$V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$.

Secondly, we are going to prove that $k > m$. Assume $k \leq m$. Then $V(\mathbf{P}_k, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) = V(\mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$. Hence $V(\mathbf{P}^\infty_m, \mathbf{P}^\infty_0) = V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$. But $V(\mathbf{P}^\infty_m, \mathbf{P}^\infty_0) \neq V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0)$ (by Lemma 3.4.12(8)).

Thirdly, we are going to prove that $k = n$. Since $k > m$, we have $\mathbf{P}^\infty_m \models \mathsf{Alt}_k \approx \mathbf{0}$. Hence $\mathbf{P}_k, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0 \models \mathsf{Alt}_k \approx \mathbf{0}$ and $V(\mathbf{P}_k, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) \models \mathsf{Alt}_k \approx \mathbf{0}$. Then $V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) \models \mathsf{Alt}_k \approx \mathbf{0}$ and so $\mathbf{P}_n \models \mathsf{Alt}_k \approx \mathbf{0}$. Therefore $n \leq k$ (by Lemma 3.4.5). Hence $k = n$.

So $S = \{\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0\}$. Thus $V(\mathbf{P}_n, \mathbf{P}^\infty_m, \mathbf{P}^\infty_0) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$. □

Fact 3.4.25. *For $1 \leq n < \omega$, $V(\mathbf{P}^\infty_n)$ (= $V(\mathbf{P}_k, \mathbf{P}^\infty_n)$ for any $1 \leq k \leq n$) is characterized by $\{\exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\mathbf{P}^\infty_n) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$. Since $\mathbf{P}^\infty_n \models \exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}$, we have $V(\mathbf{P}^\infty_n) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since $\mathbf{P}^\infty_0 \not\models \exists E \approx \mathbf{1}$, we have $\mathbf{P}^\infty_0 \notin S$.

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Then $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$. Hence $k \leq n$ (by Lemma 3.4.5). Therefore $V(\mathbf{P}^\infty_n) \subseteq V(S) = V(\mathbf{P}_k) \subseteq V(\mathbf{P}_n)$ [since $k \leq n$] $\subseteq V(\mathbf{P}^\infty_n)$ and so $V(\mathbf{P}^\infty_n) = V(\mathbf{P}_k)$. But $V(\mathbf{P}^\infty_n) \neq V(\mathbf{P}_k)$ (by Lemma 3.4.12(1)).

So it remains that either $S = \{\mathbf{P}^\infty_j\}$ (for some $1 \leq j \leq \omega$) or $S = \{\mathbf{P}_k, \mathbf{P}^\infty_j\}$ (for some $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$).

- Suppose $S = \{\mathbf{P}^\infty_j\}$ for some $1 \leq j \leq \omega$. Then $\mathbf{P}^\infty_j \models \mathsf{Alt}_n \approx \mathbf{0}$. Hence $j \leq n$ (by Lemma 3.4.5). If $j < n$, then $V(\mathbf{P}^\infty_j) \subset V(\mathbf{P}^\infty_n)$ [by Lemma 3.4.8] $\subseteq V(S) = V(\mathbf{P}^\infty_j)$ and so $V(\mathbf{P}^\infty_j) \subset V(\mathbf{P}^\infty_j)$. Therefore $j = n$. Thus $S = \{\mathbf{P}^\infty_n\}$ (in this case).

- Now suppose $S = \{\mathbf{P}_k, \mathbf{P}^\infty_j\}$ for some $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$. Then $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$ and $\mathbf{P}^\infty_j \models \mathsf{Alt}_n \approx \mathbf{0}$. Hence $k \leq n$ and $j \leq n$

(by Lemma 3.4.5). We are going to prove that $j = n$. Since $\mathbf{P}_k, \mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Then $\mathbf{P}_n^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$ (since $\mathbf{P}_n^\infty \in V(\mathbf{P}_n^\infty) \subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty)$). Therefore $n \leq j$ (by Corollary 3.4.11). Hence $j = n$. Thus $S = \{\mathbf{P}_k, \mathbf{P}_n^\infty\}$ for some $1 \leq k \leq n$ (in this case).

So either $S = \{\mathbf{P}_n^\infty\}$ or $S = \{\mathbf{P}_k, \mathbf{P}_n^\infty\}$ for some $1 \leq k \leq n$. Since $V(\mathbf{P}_n^\infty) = V(\mathbf{P}_k, \mathbf{P}_n^\infty)$ for any $1 \leq k \leq n$, we obtain that $V(\mathbf{P}_n^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$ ($= V(\mathbf{P}_k, \mathbf{P}_n^\infty)$ for any $1 \leq k \leq n$). □

Fact 3.4.26. *For $1 \leq m < n < \omega$, $V(\mathbf{P}_n, \mathbf{P}_m^\infty)$ is characterized by $\Gamma = \{\exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}, E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\mathbf{P}_n, \mathbf{P}_m^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$. Since $\mathbf{P}_n, \mathbf{P}_m^\infty \models \Gamma$, we have $V(\mathbf{P}_n, \mathbf{P}_m^\infty) \subseteq \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\} \neq V_0$. Since $\{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$ is an equational class of MBA's, $V(S) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$ where $S \neq \emptyset$ consists of at most three special MBA's.

Since $\mathbf{P}_0^\infty \not\models \exists E \approx \mathbf{1}$, we get $\mathbf{P}_0^\infty \notin S$.

Assume $S = \{\mathbf{P}_k\}$ for some $1 \leq k \leq \omega$. Then $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$. Hence $k \leq n$ (by Lemma 3.4.5). Then $V(\mathbf{P}_k) \subseteq V(\mathbf{P}_n) \subseteq V(\mathbf{P}_n, \mathbf{P}_m^\infty) \subseteq V(S) = V(\mathbf{P}_k)$ and so $V(\mathbf{P}_k) = V(\mathbf{P}_n, \mathbf{P}_m^\infty)$. But $V(\mathbf{P}_k) \neq V(\mathbf{P}_n, \mathbf{P}_m^\infty)$ (by Lemma 3.4.12(1)).

Assume $S = \{\mathbf{P}_j^\infty\}$ for some $1 \leq j \leq \omega$. Then $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}$. Hence $j \leq m$ (by Corollary 3.4.11). Therefore $V(\mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_m^\infty)$ [since $j \leq m$] $\subseteq V(\mathbf{P}_n, \mathbf{P}_m^\infty) \subseteq V(S) = V(\mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_j^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty)$. But $V(\mathbf{P}_j^\infty) \neq V(\mathbf{P}_n, \mathbf{P}_m^\infty)$ (by Lemma 3.4.12(7)).

So it remains that $S = \{\mathbf{P}_k, \mathbf{P}_j^\infty\}$ for some $1 \leq k \leq \omega$ and $1 \leq j \leq \omega$. Then $\mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0}$ and $\mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}$. Hence $k \leq n$ (by Lemma 3.4.5) and $j \leq m$ (by Corollary 3.4.11). Therefore $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \subseteq V(\mathbf{P}_n, \mathbf{P}_m^\infty)$ [since $k \leq n$ and $j \leq m$] $\subseteq V(S) = V(\mathbf{P}_k, \mathbf{P}_j^\infty)$ and so $V(\mathbf{P}_k, \mathbf{P}_j^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty)$.

We are going to prove that $j = m$. Since $\mathbf{P}_k, \mathbf{P}_j^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$, we have $V(\mathbf{P}_k, \mathbf{P}_j^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Then $V(\mathbf{P}_n, \mathbf{P}_m^\infty) \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$ and so $\mathbf{P}_m^\infty \models E \vee (\mathsf{Alt}_j)' \approx \mathbf{1}$. Hence $m \leq j$ (by Corollary 3.4.11). Thus $j = m$.

So $S = \{\mathbf{P}_k, \mathbf{P}_m^\infty\}$ and $V(\mathbf{P}_k, \mathbf{P}_m^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty)$.

If $k \leq m$, then $V(\mathbf{P}_k, \mathbf{P}_m^\infty) = V(\mathbf{P}_m^\infty)$. Hence $V(\mathbf{P}_m^\infty) = V(\mathbf{P}_n, \mathbf{P}_m^\infty)$. But $V(\mathbf{P}_m^\infty) \neq V(\mathbf{P}_n, \mathbf{P}_m^\infty)$ (by Lemma 3.4.12(7)). Thus $k > m$.

Next we are going to prove that $k = n$. Since $k > m$, we have $\mathbf{P}_k, \mathbf{P}_m^\infty \models \mathsf{Alt}_k \approx \mathbf{0}$. Hence $V(\mathbf{P}_k, \mathbf{P}_m^\infty) \models \mathsf{Alt}_k \approx \mathbf{0}$. Then $V(\mathbf{P}_n, \mathbf{P}_m^\infty) \models \mathsf{Alt}_k \approx \mathbf{0}$ and so $\mathbf{P}_n \models \mathsf{Alt}_k \approx \mathbf{0}$. Therefore $n \leq k$ (by Lemma 3.4.5). Hence $k = n$.

So $S = \{\mathbf{P}_n, \mathbf{P}_m^\infty\}$. Thus $V(\mathbf{P}_n, \mathbf{P}_m^\infty) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$. □

Fact 3.4.27. *$V(\emptyset)$ is characterized by $\{v_0 \approx v_1\}$ relative to $\bar{V}$.*

Proof. To be proved that $V(\emptyset) = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models v_0 \approx v_1\}$. Part $\subseteq$. Let $\mathbf{A} \in V(\emptyset)$. Hence $\mathbf{A} \in V_0$ (since $V(\emptyset) = V_0$). Then $\mathbf{A} \in \bar{V}$ (since $V_0 \subseteq \bar{V}$) and $\mathbf{A} \models v_0 \approx v_1$ (otherwise there exist at least two different elements in $\mathbf{A}$). Part $\supseteq$. Let $\mathbf{A} \in \bar{V}$ and $\mathbf{A} \models v_0 \approx v_1$. Then $\mathbf{A} \neq \emptyset$ (since $\mathbf{A}$ is an algebra) and $p = q$ for all $p, q \in \mathbf{A}$. So $\mathbf{A}$ is one-element. Therefore $\mathbf{A} \in V(\emptyset)$. □

Corollary 3.4.28. *Let X be an infinite set of variables and V a variety of MBA's. Then $Id_V(X)$ is finitely based.*

Proof. By Theorem 3.4.13, $V = \{\mathbf{A} \in \bar{V} \mid \mathbf{A} \models \Gamma\}$ for some finite set Γ of equations. To be proved that $\mathsf{Axioms} \cup \Gamma \models Id_V(X)$, where Axioms is the set of six equations from Definition 2.2.1. Suppose $\mathbf{A} \models \mathsf{Axioms} \cup \Gamma$. Then $\mathbf{A} \in \bar{V}$ and $\mathbf{A} \models \Gamma$. Hence $\mathbf{A} \in V$. Thus $\mathsf{Axioms} \cup \Gamma \models Id_V(X)$. So $Id_V(X)$ is finitely based. □

Chapter 4

Finitely generated MBA's

In [1] H. Bass considers finitely generated monadic algebras. The present chapter, except Section 4.2, is similar to his paper. In Section 4.1 we introduce (as in [1]) useful notations and prove that every finitely generated MBA is finite (an upper bound on the number of elements is provided). In Section 4.2 we show that the number of elements of a free MBA on a finite set achieves its upper bound. Section 4.3 states a necessary and sufficient condition under which certain maps between finitely generated MBA's can be extended to MBA-homomorphisms. In Section 4.4 we construct a free MBA on any finite set. In Section 4.1 and Section 4.4 two particular cases are considered explicitly.

4.1 Finitely generated MBA's are finite

In this section we introduce (as in [1]) useful notations and prove that every finitely generated MBA is finite (an upper bound on the number of atoms and an upper bound on the number of elements are provided). For better understanding, two particular cases of the notation are considered explicitly. In addition, several other technical results are given.

Definition 4.1.1. $D = \{-1, 1\}$.

Definition 4.1.2. *For* $0 \leq n < \omega$, $D^n = \underbrace{D \times \cdots \times D}_{n}$. *(Note that, in particular,* $D^0 = \{\emptyset\}$, *i.e.* D^0 *consists of only one zero-dimensional vector.)*

Definition 4.1.3. *For* $0 \leq n < \omega$, $i < n$ *and* $e \in D^n$, e_i *is the ith coordinate of* e.

Definition 4.1.4. *For* $0 \leq n < \omega$ *and* $i < n$, $D^n_i = \{e \in D^n \mid e_i = 1\}$.

Let $0 \leq r < \omega$ and $m = 2^r$ (cases $r = 0$ and $r = 1$ are provided below). We choose some definite enumeration $\{e^i \mid 0 \leq i < m\}$ of D^r.

Suppose $(\mathbf{M}, \wedge, \vee, ', \mathbf{0}, \mathbf{1}, E, \exists)$ is an MBA and $\{p_0, \ldots, p_{r-1}\} \subseteq \mathbf{M}$.

Definition 4.1.5. *For* $p \in \mathbf{M}$, $p^1 = p$ *and* $p^{-1} = p'$.

Definition 4.1.6. $P = (p_0, \ldots, p_{r-1})$.

Definition 4.1.7. *For* $e \in D^r$, $\lambda_P(e) = \bigwedge_{i<r} p_i^{e_i}$.

Definition 4.1.8. $\lambda P = (\lambda_P(e^0), \ldots, \lambda_P(e^{m-1}))$.

Definition 4.1.9. $\exists\lambda P = (\exists\lambda_P(e^0), \ldots, \exists\lambda_P(e^{m-1}))$.

Definition 4.1.10. *For* $d \in D^m$, $\lambda_{\exists\lambda P}(d) = \bigwedge_{i<m}(\exists\lambda_P(e^i))^{d_i}$.

Definition 4.1.11. *For* $i < m$ *and* $d \in D^m_i$, $\mu_{\lambda P,i}(d) = \lambda_P(e^i) \wedge E \wedge \lambda_{\exists\lambda P}(d)$.

Definition 4.1.12. *For* $i < m$ *and* $d \in D^m$, $\eta_{\lambda P,i}(d) = \lambda_P(e^i) \wedge E' \wedge \lambda_{\exists\lambda P}(d)$.

Note that $P, \lambda P, \exists\lambda P$ are vectors, whereas $\lambda_P(e), \lambda_{\exists\lambda P}(d), \mu_{\lambda P,i}(d), \eta_{\lambda P,i}(d)$ are elements in $\mathbf{M}$.

We are going to look at particular cases.

Let $r = 0$. Then

- $m = 1$;
- $D^r = D^0 = \{\emptyset\}$ and so enumerate $D^0 = \{e^0\}$ where $e^0 = \emptyset$;
- $D^m = D^1 = \{(1), (-1)\}$ and $D^m_0 = D^1_0 = \{(1)\}$;
- $P = \emptyset$ (zero-dimensional vector);

- if $e \in D^r(= D^0)$, then $e = e^0$ and so $\lambda_P(e) = \lambda_P(e^0) = \lambda_\emptyset(\emptyset) = \mathbf{1}$ (since $\bigwedge \emptyset = \mathbf{1}$);
- $\lambda P = (\lambda_P(e^0)) = (\lambda_\emptyset(\emptyset)) = (\mathbf{1})$ (one dimensional vector whose single coordinate is the unit element of **M**);
- $\exists \lambda P = (\exists \mathbf{1})$ (one dimensional vector);
- if $d \in D^m(= D^1)$, then either $d = (1)$ or $d = (-1)$; so if $d \in D^m(= D^1)$, then either
  - $\lambda_{\exists \lambda P}(d) = \lambda_{(\exists \mathbf{1})}((1)) = (\exists \mathbf{1})^1 = \exists \mathbf{1}$ or
  - $\lambda_{\exists \lambda P}(d) = \lambda_{(\exists \mathbf{1})}((-1)) = (\exists \mathbf{1})^{-1} = (\exists \mathbf{1})'$;
- if $i < m(= 1)$, then $i = 0$; if $d \in D_0^m(= D_0^1)$, then $d = (1)$; so if $i < m(= 1)$ and $d \in D_i^m(= D_0^m)$, then there is only one μ-expression $\mu_{\lambda P, i}(d) = \mu_{(\mathbf{1}),0}((1)) = \lambda_P(e^0) \wedge E \wedge \lambda_{(\exists \mathbf{1})}((1)) = \mathbf{1} \wedge E \wedge (\exists \mathbf{1})^1 = E \wedge \exists \mathbf{1}$ (although it is equal to E, it is better to keep it in this form);
- if $i < m(= 1)$, then $i = 0$; if $d \in D^m(= D^1)$, then either $d = (1)$ or $d = (-1)$; so if $i < m$ and $d \in D^m$, then either
  - $\eta_{\lambda P, i}(d) = \eta_{(\mathbf{1}),0}((1)) = \lambda_P(e^0) \wedge E' \wedge \lambda_{(\exists \mathbf{1})}((1)) = \mathbf{1} \wedge E' \wedge (\exists \mathbf{1})^1 = E' \wedge \exists \mathbf{1}$ or
  - $\eta_{\lambda P, i}(d) = \eta_{(\mathbf{1}),0}((-1)) = \lambda_P(e^0) \wedge E' \wedge \lambda_{(\exists \mathbf{1})}((-1)) = \mathbf{1} \wedge E' \wedge (\exists \mathbf{1})^{-1} = E' \wedge (\exists \mathbf{1})'$.

So there are one μ-expression and two η-expressions in this case.

Now let $r = 1$. Then

- $m = 2$;
- $D^r = D^1 = \{(1), (-1)\}$ and so enumerate $D^1 = \{e^0, e^1\}$ where $e^0 = (1)$, $e^1 = (-1)$ (one dimensional vectors);

- $D^m = D^2 = \{(1,1),(-1,1),(1,-1),(-1,-1)\}$, $D_0^2 = \{(1,1),(1,-1)\}$ and $D_1^2 = \{(1,1),(-1,1)\}$;
- $P = (p_0)$ (one dimensional vector);
- if $e \in D^r (= D^1)$, then either $e = e^0$ or $e = e^1$; so if $e \in D^r$, then either
  - $\lambda_P(e) = \lambda_P(e^0) = \lambda_{(p_0)}(e^0) = \bigwedge_{i<1} p_i^{e_i^0} = p_0^{e_0^0} = p_0^1 = p_0$ or
  - $\lambda_P(e) = \lambda_P(e^1) = \lambda_{(p_0)}(e^1) = \bigwedge_{i<1} p_i^{e_i^1} = p_0^{e_0^1} = p_0^{-1} = p_0'$;
- $\lambda P = (\lambda_P(e^0), \lambda_P(e^1)) = (p_0, p_0')$;
- $\exists\lambda P = (\exists p_0, \exists(p_0'))$;
- if $d \in D^m (= D^2)$, then one of the following holds:
  - $\lambda_{\exists\lambda P}(d) = \lambda_{(\exists p_0, \exists(p_0'))}((1,1)) = (\exists p_0)^1 \wedge (\exists(p_0'))^1 = \exists p_0 \wedge \exists(p_0')$,
  - $\lambda_{\exists\lambda P}(d) = \lambda_{(\exists p_0, \exists(p_0'))}((-1,1)) = (\exists p_0)^{-1} \wedge (\exists(p_0'))^1 = (\exists p_0)' \wedge \exists(p_0')$,
  - $\lambda_{\exists\lambda P}(d) = \lambda_{(\exists p_0, \exists(p_0'))}((1,-1)) = (\exists p_0)^1 \wedge (\exists(p_0'))^{-1} = \exists p_0 \wedge (\exists(p_0'))'$,
  - $\lambda_{\exists\lambda P}(d) = \lambda_{(\exists p_0, \exists(p_0'))}((-1,-1)) = (\exists p_0)^{-1} \wedge (\exists(p_0'))^{-1} = (\exists p_0)' \wedge (\exists(p_0'))'$;
- if $i < m (= 2)$, then either $i = 0$ or $i = 1$; if $i = 0$ and $d \in D_i^m (= D_0^2)$, then either
  - $\mu_{\lambda P, i}(d) = \mu_{(p_0, p_0'),0}((1,1)) = \lambda_P(e^0) \wedge E \wedge \lambda_{(\exists p_0, \exists(p_0'))}((1,1)) = p_0 \wedge E \wedge \exists p_0 \wedge \exists(p_0')$ or
  - $\mu_{\lambda P, i}(d) = \mu_{(p_0, p_0'),0}((1,-1)) = \lambda_P(e^0) \wedge E \wedge \lambda_{(\exists p_0, \exists(p_0'))}((1,-1)) = p_0 \wedge E \wedge \exists p_0 \wedge (\exists(p_0'))'$;

  if $i = 1$ and $d \in D_i^m (= D_1^2)$, then either
  - $\mu_{\lambda P, i}(d) = \mu_{(p_0, p_0'),1}((1,1)) = \lambda_P(e^1) \wedge E \wedge \lambda_{(\exists p_0, \exists(p_0'))}((1,1)) = p_0' \wedge E \wedge \exists p_0 \wedge \exists(p_0')$ or

- $\mu_{\lambda P,i}(d) = \mu_{(p_0,p_0'),1}((-1,1)) = \lambda_P(e^1) \wedge E \wedge \lambda_{(\exists p_0,\exists(p_0'))}((-1,1)) = p_0' \wedge E \wedge (\exists p_0)' \wedge \exists(p_0')$;

- if $i < m(= 2)$, then either $i = 0$ or $i = 1$; if $i = 0$ and $d \in D^m(= D^2)$, then one the following holds:
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),0}((1,1)) = \lambda_P(e^0) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((1,1)) = p_0 \wedge E' \wedge \exists p_0 \wedge \exists(p_0')$,
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),0}((-1,1)) = \lambda_P(e^0) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((-1,1)) = p_0 \wedge E' \wedge (\exists p_0)' \wedge \exists(p_0')$,
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),0}((1,-1)) = \lambda_P(e^0) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((1,-1)) = p_0 \wedge E' \wedge \exists p_0 \wedge (\exists(p_0'))'$,
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),0}((-1,-1)) = \lambda_P(e^0) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((-1,-1)) = p_0 \wedge E' \wedge (\exists p_0)' \wedge (\exists(p_0'))'$;

  if $i = 1$ and $d \in D^m(= D^2)$, then one of the following holds:
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),1}((1,1)) = \lambda_P(e^1) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((1,1)) = p_0' \wedge E' \wedge \exists p_0 \wedge \exists(p_0')$,
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),1}((-1,1)) = \lambda_P(e^1) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((-1,1)) = p_0' \wedge E' \wedge (\exists p_0)' \wedge \exists(p_0')$,
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),1}((1,-1)) = \lambda_P(e^1) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((1,-1)) = p_0' \wedge E' \wedge \exists p_0 \wedge (\exists(p_0'))'$,
  - $\eta_{\lambda P,i}(d) = \eta_{(p_0,p_0'),1}((-1,-1)) = \lambda_P(e^1) \wedge E' \wedge \lambda_{(\exists p_0,\exists(p_0'))}((-1,-1)) = p_0' \wedge E' \wedge (\exists p_0)' \wedge (\exists(p_0'))'$.

So there are four μ-expressions and eight η-expressions in this case.

Definition 4.1.13. *An MBA* $(\mathbf{M}, E, \exists)$ *is* ***free on a set of generators*** $G \subseteq \mathbf{M}$ *(or,* $(\mathbf{M}, E, \exists)$ *is* ***freely generated by a subset*** $G \subseteq \mathbf{M}$*) iff (i) G generates* $\mathbf{M}$*, and (ii) any map* f_0 *of G into an MBA* $(\mathbf{A}, E, \exists)$ *can be extended to an MBA-homomorphism* $f : \mathbf{M} \to \mathbf{A}$.

Definition 4.1.14. *Let* $\mathbf{B}$ *be a Boolean algebra and* $\sigma = \{p_0, \ldots, p_{n-1}\}$ *be a finite subset of* $\mathbf{B}$. σ *is a* ***partition*** *of* $p \in \mathbf{B}$ *iff*

- $p_i \wedge p_j = \mathbf{0}$ *for* $i \neq j$ *and*
- $\bigvee_{i<n} p_i = p$.

For future reference, we state some facts from the theory of Boolean algebras (see [1]).

Lemma 4.1.15. *Every finitely generated Boolean algebra is finite.*

Lemma 4.1.16 (see [1, (1.1)])**.** *If* $p_0, \ldots, p_{n-1}$ *generate a Boolean algebra* $\mathbf{B}$*, then* $\{\lambda_P(e) \mid e \in D^n\}$ *(where* $P = (p_0, \ldots, p_{n-1})$*) is a partition of* $\mathbf{1}$ *whose nonzero elements are just the atoms of* $\mathbf{B}$.

Suppose $(\mathbf{M}, E, \exists)$ is an MBA, $0 \leq r < \omega$, $m = 2^r$ and $\{p_0, \ldots, p_{r-1}\} \subseteq \mathbf{M}$. Put $P = (p_0, \ldots, p_{r-1})$.

Lemma 4.1.17. *For* $i < m$ *and* $d \in D_i^m$, $\exists(\mu_{\lambda P,i}(d)) = \lambda_{\exists\lambda P}(d)$.

Proof.

$$\begin{aligned}
\exists(\mu_{\lambda P,i}(d)) &= \exists\big(\lambda_P(e^i) \wedge E \wedge \lambda_{\exists\lambda P}(d)\big) \\
&= \exists\Bigg(\lambda_P(e^i) \wedge E \wedge \Big(\bigwedge_{j<m} (\exists\lambda_P(e^j))^{d_j}\Big)\Bigg) \\
&= \exists\big(\lambda_P(e^i) \wedge E\big) \wedge \Big(\bigwedge_{j<m} (\exists\lambda_P(e^j))^{d_j}\Big) \\
&\quad \text{[by Definition 2.2.1(3) and Lemma 2.2.6(13)]} \\
&= \exists\lambda_P(e^i) \wedge \Big(\bigwedge_{j<m} (\exists\lambda_P(e^j))^{d_j}\Big) \text{ [by Definition 2.2.1(6)]} \\
&= \bigwedge_{j<m} (\exists\lambda_P(e^j))^{d_j} \text{ [since } d_i = 1] \\
&= \lambda_{\exists\lambda P}(d) \text{ [by Definition 4.1.10]}.
\end{aligned}$$

□

Lemma 4.1.18. *For $i < m$ and $d \in D^m$, $\exists(\eta_{\lambda P,i}(d)) = \mathbf{0}$.*

Proof. $\exists(\eta_{\lambda P,i}(d)) = \exists\big(\lambda_P(e^i) \wedge E' \wedge \lambda_{\exists\lambda P}(d)\big) = \mathbf{0}$ [by Lemma 2.2.6(4)]. □

Now suppose that $(\mathbf{M}, E, \exists)$ is an MBA generated by $\{p_0, \ldots, p_{r-1}\} \subseteq \mathbf{M}$. Let $\mathbf{A}_0$ be the Boolean subalgebra of $\mathbf{M}$ generated by

$$\{\lambda_P(e^k) \wedge E, \lambda_P(e^k) \wedge E', \exists\lambda_P(e^k) \mid k < m\}.$$

So the Boolean algebra $\mathbf{A}_0$ is finite (by Lemma 4.1.15) and $\mathbf{A}_0 \subseteq \mathbf{M}$. By proving that $\mathbf{M}$ and $\mathbf{A}_0$ are equal as sets, we can conclude that every finitely generated MBA is finite.

Lemma 4.1.19. *$p_0, \ldots, p_{r-1}, E$ are in $\mathbf{A}_0$.*

Proof. To be proved that $p_0 \in \mathbf{A}_0$. p_0 can be represented as follows

$$\begin{aligned}
p_0 &= p_0 \wedge \mathbf{1} = p_0 \wedge ((p_0 \wedge \cdots \wedge p_{r-1} \wedge E) \vee (p_0 \wedge \cdots \wedge p_{r-1} \wedge E)') \\
&= p_0 \wedge ((p_0 \wedge \cdots \wedge p_{r-1} \wedge E) \vee (p_0' \vee \cdots \vee p_{r-1}' \vee E')) \\
&= (p_0 \wedge \cdots \wedge p_{r-1} \wedge E) \vee \\
&\vee ((p_0 \wedge p_0') \vee (p_0 \wedge p_2') \vee \cdots \vee (p_0 \wedge p_{r-1}') \vee (p_0 \wedge E')) \\
&= (p_0 \wedge \cdots \wedge p_{r-1} \wedge E) \vee \\
&\vee ((p_0 \wedge p_1') \vee (p_0 \wedge p_2') \vee \cdots \vee (p_0 \wedge p_{r-1}') \vee (p_0 \wedge E')).
\end{aligned}$$

Using the fact that $a = (a \wedge b) \vee (a \wedge b')$, we obtain that

$$\begin{aligned}
p_0 \wedge p_1' &= \bigvee_{e_2, e_3, \ldots, e_{r-1}, e_r \in \{\pm 1\}} \left(p_0 \wedge p_1' \wedge p_2^{e_2} \wedge p_3^{e_3} \wedge \cdots \wedge p_{r-1}^{e_{r-1}} \wedge E^{e_r}\right), \\
p_0 \wedge p_2' &= \bigvee_{e_1, e_3, \ldots, e_{r-1}, e_r \in \{\pm 1\}} \left(p_0 \wedge p_1^{e_1} \wedge p_2' \wedge p_3^{e_3} \wedge \cdots \wedge p_{r-1}^{e_{r-1}} \wedge E^{e_r}\right), \\
&\cdots \\
p_0 \wedge p_{r-1}' &= \bigvee_{e_1, e_2, e_3, \ldots, e_{r-2}, e_r \in \{\pm 1\}} \left(p_0 \wedge p_1^{e_1} \wedge p_e^{e_2} \wedge p_3^{e_3} \wedge \cdots \wedge p_{r-2}^{e_{r-2}} \wedge p_{r-1}' \wedge E^{e_r}\right), \\
p_0 \wedge E' &= \bigvee_{e_1, e_2, \ldots, e_{r-2}, e_{r-1} \in \{\pm 1\}} \left(p_0 \wedge p_1^{e_1} \wedge p_2^{e_2} \wedge \cdots \wedge p_{r-2}^{e_{r-2}} \wedge p_{r-1}^{e_{r-1}} \wedge E'\right).
\end{aligned}$$

So $p_0 \wedge p'_1, \ldots, p_0 \wedge p'_{r-1}, p_0 \wedge E'$ are finite Boolean combinations of some elements in $\{\lambda_P(e^k) \wedge E, \lambda_P(e^k) \wedge E', \exists\lambda_P(e^k) \mid k < m\}$. By substituting them into the representation of p_0, we get that p_0 belongs to $\mathbf{A}_0$ (since $\mathbf{A}_0$ is closed under finite Boolean combinations).

Similarly it is possible to prove that $p_1, \ldots, p_{r-1}, E \in \mathbf{A}_0$. □

Lemma 4.1.20. $\{\lambda_P(e^k) \wedge E, \lambda_P(e^k) \wedge E' \mid k < m\}$ *is a partition of* $\mathbf{1}$.

Proof. Since the collection consists of all possible combinations

$$\textstyle\bigwedge_{i<r} p_i^{\pm} \wedge E^{\pm},$$

the collection is a partition of $\mathbf{1}$. □

Lemma 4.1.21.

$$\left\{\left(\bigwedge_{i<m}(\lambda_P(e^i)\wedge E)^{c_i}\right)\wedge\left(\bigwedge_{i<m}(\lambda_P(e^i)\wedge E')^{\bar{c}_i}\right)\wedge\left(\bigwedge_{i<m}(\exists\lambda_P(e^i))^{d_i}\right)\middle| c, \bar{c}, d \in D^m\right\}$$

is a partition of $\mathbf{1}$ *whose nonzero elements are just the atoms of the Boolean algebra* $\mathbf{A}_0$.

Proof. Since the set $\{\lambda_P(e^k) \wedge E, \lambda_P(e^k) \wedge E', \exists\lambda_P(e^k) \mid k < m\}$ generates the Boolean algebra $\mathbf{A}_0$, we just apply Lemma 4.1.16. □

Lemma 4.1.22. *Every atom of the Boolean algebra* $\mathbf{A}_0$ *is equal to either*

- $\mu_{\lambda P,j}(d)$, *for some* $j < m$ *and* $d \in D_j^m$, *or*
- $\eta_{\lambda P,j}(d)$, *for some* $j < m$ *and* $d \in D^m$.

Proof. Suppose $a \in \mathbf{A}_0$ is an atom. Hence $a \neq \mathbf{0}$. By Lemma 4.1.21,

$$a = \left(\bigwedge_{i<m}(\lambda_P(e^i) \wedge E)^{c_i}\right)\wedge\left(\bigwedge_{i<m}(\lambda_P(e^i) \wedge E')^{\bar{c}_i}\right)\wedge\left(\bigwedge_{i<m}(\exists\lambda_P(e^i))^{d_i}\right) \quad (4.1.1)$$

for some $c, \bar{c}, d \in D^m$.

To be proved that precisely one element in $\{c_0, \ldots, c_{m-1}, \bar{c}_0, \ldots, \bar{c}_{m-1}\}$ is equal to 1. Since $a \neq \mathbf{0}$, we have

$$\left(\bigwedge_{i<m} (\lambda_P(e^i) \wedge E)^{c_i}\right) \wedge \left(\bigwedge_{i<m} (\lambda_P(e^i) \wedge E')^{\bar{c}_i}\right) \neq \mathbf{0}. \tag{4.1.2}$$

If every element in $\{c_0, \ldots, c_{m-1}, \bar{c}_0, \ldots, \bar{c}_{m-1}\}$ were equal to -1, then we would obtain

$$\begin{aligned}
&\left(\bigwedge_{i<m} (\lambda_P(e^i) \wedge E)^{c_i}\right) \wedge \left(\bigwedge_{i<m} (\lambda_P(e^i) \wedge E')^{\bar{c}_i}\right) \\
&\qquad = \left(\bigwedge_{i<m} (\lambda_P(e^i) \wedge E)^{-1}\right) \wedge \left(\bigwedge_{i<m} (\lambda_P(e^i) \wedge E')^{-1}\right) \\
&\qquad = \left(\left(\bigvee_{i<m} (\lambda_P(e^i) \wedge E)\right) \vee \left(\bigvee_{i<m} (\lambda_P(e^i) \wedge E')\right)\right)^{-1} \\
&\qquad = \mathbf{1}' \text{ [by Lemma 4.1.20]} \\
&\qquad = \mathbf{0}.
\end{aligned}$$

Therefore at least one element in $\{c_0, \ldots, c_{m-1}, \bar{c}_0, \ldots, \bar{c}_{m-1}\}$ is equal to 1. Similarly (in particular, using Lemma 4.1.20) it is possible to prove that such an element is the only one.

So precisely one element in $\{c_0, \ldots, c_{m-1}, \bar{c}_0, \ldots, \bar{c}_{m-1}\}$ is equal to 1. There are two cases:

Case 1 $c_j = 1$ for some $j < m$ (and others are -1). Using Lemma 4.1.20 and the fact $p \wedge q = \mathbf{0}$ iff $p \leq q'$ iff $p \wedge q' = p$, we obtain

$$a = \lambda_P(e^j) \wedge E \wedge \left(\bigwedge_{i<m} (\exists \lambda_P(e^i))^{d_i}\right). \tag{4.1.3}$$

It follows from $a \neq \mathbf{0}$ and $\lambda_P(e^j) \wedge E \leq \exists \lambda_P(e^j)$ that $d \in D_j^m$. Thus $a = \mu_{\lambda P,j}(d)$ and $d \in D_j^m$.

Case 2 $\bar{c}_j = 1$ for some $j < m$ (and others are -1). Analogously,

$$a = \lambda_P(e^j) \wedge E' \wedge \left(\bigwedge_{i<m} (\exists \lambda_P(e^i))^{d_i}\right). \tag{4.1.4}$$

Thus $a = \eta_{\lambda P,j}(d)$ and $d \in D^m$.

□

Lemma 4.1.23. *If $a \in \mathbf{A}_0$ is an atom, then $\exists a \in \mathbf{A}_0$.*

Proof. By Lemma 4.1.22, there are two cases.

If $a = \mu_{\lambda P,j}(d)$ for some $j < m$ and $d \in D_j^m$, then $\exists a = \exists(\mu_{\lambda P,j}(d)) = \lambda_{\exists\lambda P}(d)$ [by Lemma 4.1.17] $= \bigwedge_{i<m}(\exists\lambda_P(e^i))^{d_i} \in \mathbf{A}_0$ [by definition of $\mathbf{A}_0$].

If $a = \eta_{\lambda P,j}(d)$ for some $j < m$ and $d \in D^m$, then $\exists a = \exists(\eta_{\lambda P,j}(d)) = \mathbf{0}$ [by Lemma 4.1.18] $\in \mathbf{A}_0$. □

Lemma 4.1.24. *The Boolean algebra $\mathbf{A}_0$ is closed under $\exists$, i.e. $\exists p \in \mathbf{A}_0$ for all $p \in \mathbf{A}_0$.*

Proof. Suppose $p \in \mathbf{A}_0$. Let $\{a_0, \ldots, a_{l-1}\}$ be the set of all atoms of $\mathbf{A}_0$ such that $a_i \leq p$ for $i < l$. Since $\mathbf{A}_0$ is a finite Boolean algebra, $p = \bigvee_{i<l} a_i$, and so $\exists p = \exists\left(\bigvee_{i<l} a_i\right) = \bigvee_{i<l} \exists a_i \in \mathbf{A}_0$ (by Lemma 4.1.23). □

Theorem 4.1.25. *The MBA $(\mathbf{M}, E, \exists)$ is finite.*

Proof. It follows from Lemma 4.1.19 and Lemma 4.1.24 that $\mathbf{M} = \mathbf{A}_0$ as sets. Since $\mathbf{A}_0$ is finite, we conlude that $\mathbf{M}$ is finite. □

Theorem 4.1.26. *The MBA $(\mathbf{M}, E, \exists)$ has at most $3 \cdot 2^r \cdot 2^{2^r-1}$ atoms.*

Proof. Let $a \in \mathbf{M}$ be an atom. Hence a is an atom of $\mathbf{A}_0$ (since $\mathbf{M} = \mathbf{A}_0$ as sets). Then, by Lemma 4.1.22, either $a = \mu_{\lambda P,j}(d)$, for some $j < m$ and $d \in D_j^m$, or $a = \eta_{\lambda P,j}(d)$, for some $j < m$ and $d \in D^m$. So it suffices to count the number of elements of the set

$$\{\mu_{\lambda P,j}(d) \mid j < m, d \in D_j^m\} \cup \{\eta_{\lambda P,j}(d) \mid j < m, d \in D^m\}.$$

Thus there are at most

$$m \cdot 2^{m-1} + m \cdot 2^m = (1+2) \cdot m \cdot 2^{m-1} = 3 \cdot 2^r \cdot 2^{2^r-1}$$

atoms in $(\mathbf{M}, E, \exists)$. □

Since every element in $\mathbf{M}$ is a supremum of some finite set of atoms, we have

Corollary 4.1.27. *There are at most* $2^{3\cdot 2^r\cdot 2^{2^r-1}}$ *elements in* $(\mathbf{M}, E, \exists)$.

In particular, an MBA generated by the empty set has at most 3 atoms and 8 elements and an MBA generated by one element has at most 12 atoms and $2^{12} = 4096$ elements.

Let us give other results which will be useful in the following sections. Define

$$\Omega = \{\mu_{\lambda P,i}(d) \mid i < m, d \in D_i^m\} \cup \{\eta_{\lambda P,i}(d) \mid i < m, d \in D^m\}. \qquad (4.1.5)$$

So Ω is a partition of $\mathbf{1}$ whose nonzero elements are just the atoms of $(\mathbf{M}, E, \exists)$.

Lemma 4.1.28. *For every* $d \in D^m$,

$$\lambda_{\exists\lambda P}(d) = \bigvee_{\{i|d_i=1\},k<m} \{\mu_{\lambda P,i}(d), \eta_{\lambda P,k}(d)\}. \qquad (4.1.6)$$

Proof. Let $d \in D^m$ be fixed.

If $d_i = 1$ for some $i < m$, then $d \in D_i^m$ (hence $\mu_{\lambda P,i}(d)$ is defined) and $\mu_{\lambda P,i}(d) = \lambda_P(e^i) \wedge E \wedge \lambda_{\exists\lambda P}(d) \le \lambda_{\exists\lambda P}(d)$. For every $k < m$, $\eta_{\lambda P,k}(d) = \lambda_P(e^k) \wedge E' \wedge \lambda_{\exists\lambda P}(d) \le \lambda_{\exists\lambda P}(d)$.

Next to be proved that the other members of Ω are disjoint from $\lambda_{\exists\lambda P}(d)$. Suppose $c \in D_j^m$ and $c \ne d$ (j may be equal to some i with $d_i = 1$). Then $\lambda_{\exists\lambda P}(d) \wedge \mu_{\lambda P,j}(c) = \lambda_{\exists\lambda P}(d) \wedge \lambda_P(e^j) \wedge E \wedge \lambda_{\exists\lambda P}(c) = \mathbf{0}$ (since $d \ne c$). Suppose $c \in D^m$ and $c \ne d$. Then, for every $k < m$, $\lambda_{\exists\lambda P}(d) \wedge \eta_{\lambda P,k}(c) = \lambda_{\exists\lambda P}(d) \wedge \lambda_P(e^k) \wedge E' \wedge \lambda_{\exists\lambda P}(c) = \mathbf{0}$ (since $d \ne c$).

Thus we have proved (4.1.6) (since every element in $(\mathbf{M}, E, \exists)$ is the supremum of atoms it contains). □

Corollary 4.1.29. *For every* $i < m$ *and* $d \in D_i^m$,

$$\exists(\mu_{\lambda P,i}(d)) = \bigvee_{\{j|d_j=1\},k<m} \{\mu_{\lambda P,j}(d), \eta_{\lambda P,k}(d)\}. \qquad (4.1.7)$$

Proof. Follows from Lemma 4.1.17 and Lemma 4.1.28. □

Lemma 4.1.30. *For $i < m$,*

$$\lambda_P(e^i) = \bigvee_{d \in D_i^m,\, c \in D^m} \{\mu_{\lambda P,i}(d), \eta_{\lambda P,i}(c)\}. \tag{4.1.8}$$

Proof. Let $i < m$ be fixed.

If $d \in D_i^m$, then $\mu_{\lambda P,i}(d) = \lambda_P(e^i) \wedge E \wedge \lambda_{\exists\lambda P}(d) \leq \lambda_P(e^i)$. If $c \in D^m$, then $\eta_{\lambda P,i}(c) = \lambda_P(e^i) \wedge E' \wedge \lambda_{\exists\lambda P}(c) \leq \lambda_P(e^i)$.

Next to be proved that the other members of Ω are disjoint from $\lambda_P(e^i)$. Suppose $j < m$ and $j \neq i$. Hence $e^i \neq e^j$. If $d \in D_j^m$, then $\lambda_P(e^i) \wedge \mu_{\lambda P,j}(d) = \lambda_P(e^i) \wedge \lambda_P(e^j) \wedge E \wedge \lambda_{\exists\lambda P}(d) = \mathbf{0}$ (since $e^i \neq e^j$). If $c \in D^m$, then $\lambda_P(e^i) \wedge \eta_{\lambda P,j}(c) = \lambda_P(e^i) \wedge \lambda_P(e^j) \wedge E' \wedge \lambda_{\exists\lambda P}(c) = \mathbf{0}$ (since $e^i \neq e^j$).

Thus we have proved (4.1.8). □

Lemma 4.1.31.

$$E = \bigvee_{i<m, d \in D_i^m} \mu_{\lambda P,i}(d). \tag{4.1.9}$$

Proof. For every $i < m$ and $d \in D_i^m$, $\mu_{\lambda P,i}(d) = \lambda_P(e^i) \wedge E \wedge \lambda_{\exists\lambda P}(d) \leq E$. On the other hand, for every $i < m$ and $d \in D^m$, $E \wedge \eta_{\lambda P,i}(d) = E \wedge \lambda_P(e^i) \wedge E' \wedge \lambda_{\exists\lambda P}(d) = \mathbf{0}$. □

4.2 On the number of elements of a free MBA on a finite set

The present section is concerned with the number of atoms and elements of free MBA's on finite sets. It is proved that the number of atoms and the number of elements of a free MBA on a finite set achieves its upper bound (see Theorem 4.1.26 and Corollary 4.1.27). The proof is based on the properties of free MBA's and does not require construction of the free MBA's as such.

Let $(\mathrm{M}, E, \exists)$ be a free MBA on the set $G = \{p_0, \ldots, p_{r-1}\} \subseteq \mathrm{M}$. As usual, let $m = 2^r$, $D^r = \{e^0, \ldots, e^{m-1}\}$, $P = (p_0, \ldots, p_{r-1})$.

Recall that

$$\Omega = \{\mu_{\lambda P,i}(d) \mid i < m, d \in D_i^m\} \cup \{\eta_{\lambda P,i}(d) \mid i < m, d \in D^m\}$$

is a partition of $\mathbf{1}$ whose nonzero elements are just the atoms of $(\mathbf{M}, E, \exists)$. By showing that each element in Ω is nonzero we will be able to prove that there are exactly $3 \cdot 2^r \cdot 2^{2^r-1}$ atoms in $\mathbf{M}$ (and so $\mathbf{M}$ has exactly $2^{3 \cdot 2^r \cdot 2^{2^r-1}}$ elements).

Lemma 4.2.1. *For every $i < m$ and $d \in D_i^m$, $\mu_{\lambda P,i}(d) \neq \mathbf{0}$.*

Proof. Let W be a set consisting of m elements. Then it is possible to choose $q_0, \ldots, q_{r-1} \subseteq W$ such that $\{q_0, \ldots, q_{r-1}\}$ freely generates the Boolean algebra $(\mathcal{P}(W), \cap, \cup, -, \mathbf{0}, \mathbf{1})$ of all subsets of W. Therefore, for every $e \in D^r$, $q_0^{e_0} \cap \cdots \cap q_{r-1}^{e_{r-1}} = \{w\}$ for precisely one $w \in W$. So we enumerate $W = \{w_0, \ldots, w_{m-1}\}$ according to $\{w_i\} = q_0^{e_0^i} \cap \cdots \cap q_{r-1}^{e_{r-1}^i}$ $(i < m)$.

Let $i < m$ and $d \in D_i^m$ be fixed. Our goal is to prove that $\mu_{\lambda P,i}(d) \neq \mathbf{0}$.

Define a marked directed graph $\mathcal{F}_{i,d} = (W_{i,d}, R_{i,d}, E_{i,d})$ by

$$W_{i,d} = W, E_{i,d} = \{w_j \mid d_j = 1, j < m\}, R_{i,d} = E_{i,d} \times E_{i,d}. \tag{4.2.1}$$

Since $d \in D_i^m$, we have $w_i \in E_{i,d}$. Obviously, $\mathcal{F}_{i,d}$ is a bounded graph. Therefore the complex algebra $\mathbf{P}_{\mathcal{F}_{i,d}}$ is an MBA (by Lemma 2.2.11).

Let $Q = (q_0, \ldots, q_{r-1})$.

To be proved that $\mu_{\lambda Q,i}(d) \neq \mathbf{0}$ (here we work in $\mathbf{P}_{\mathcal{F}_{i,d}}$). Consider

$$\begin{aligned} \mu_{\lambda Q,i}(d) = q_0^{e_0^i} \cap \cdots \cap q_{r-1}^{e_{r-1}^i} \cap E_{i,d} \cap \left(\langle R_{i,d}\rangle \left(q_0^{e_0^0} \cap \cdots \cap q_{r-1}^{e_{r-1}^0}\right)\right)^{d_0} \cap \ldots \\ \cdots \cap \left(\langle R_{i,d}\rangle \left(q_0^{e_0^{m-1}} \cap \cdots \cap q_{r-1}^{e_{r-1}^{m-1}}\right)\right)^{d_{m-1}} \\ = \{w_i\} \cap (\langle R_{i,d})\rangle\{w_0\})^{d_0} \cap \cdots \cap (\langle R_{i,d}\rangle\{w_{m-1}\})^{d_{m-1}}. \end{aligned}$$

We are going to prove that the whole expression is equal to $\{w_i\}$. Consider d_j for $j < m$. There are two cases:

- If $d_j = 1$, then $w_j \in E_{i,d} \Rightarrow w_i R_{i,d} w_j$ [since $w_i \in E_{i,d}$] $\Rightarrow w_i \in (\langle R_{i,d}\rangle\{w_j\}) \Rightarrow w_i \in (\langle R_{i,d}\rangle\{w_j\})^{d_j}$.

- If $d_j = -1$, then $w_j \notin E_{i,d} \Rightarrow \langle w_i, w_j \rangle \notin R_{i,d} \Rightarrow w_i \notin (\langle R_{i,d} \rangle \{w_j\}) \Rightarrow w_i \in (\langle R_{i,d} \rangle \{w_j\})^{d_j}$.

Thus $\{w_i\} \cap (\langle R_{i,d} \rangle) \{w_j\})^{d_j} = \{w_i\}$. So $\mu_{\lambda Q,i}(d) = \{w_i\}$. Hence $\mu_{\lambda Q,i}(d) \neq \mathbf{0}$.

Now define a mapping $f_0 : G \to \{q_0, \ldots, q_{r-1}\}$ by $f_0(p_k) = q_k$ for every $k < r$. Then there is an MBA-homomorphism $f : \mathbf{M} \to \mathbf{P}_{\mathcal{F}_{i,d}}$ which extends f_0 (since $\mathbf{M}$ is free on the set G of generators). Therefore $\mu_{\lambda P,i}(d) \neq \mathbf{0}$ otherwise $\mu_{\lambda Q,i}(d) = f(\mu_{\lambda P,i}(d)) = f(\mathbf{0}) = \mathbf{0}$. □

Lemma 4.2.2. *For every $i < m$ and $d \in D^m$, $\eta_{\lambda P,i}(d) \neq \mathbf{0}$.*

Proof. Let W and $q_0, \ldots, q_{r-1}$ be as in previous lemma (see the first paragraph of its proof). Then let $U = \{u_0, v_0, \ldots, u_{m-1}, v_{m-1}\}$. Define a mapping $g : U \to W$ by $g(u_i) = g(v_i) = w_i$ for every $i < m$.

Obviously, g is surjective. Then the mapping $h_g : \mathcal{P}(W) \to \mathcal{P}(U)$ is a (injective) Boolean algebra homomorphism, where $h_g(X) = g^{-1}(X)$ (cf. Definition 3.1.25).

Note that $h_g(\{w_i\}) = \{u_i, v_i\}$ and $h_g(\{w_i\}) = h_g\left(q_0^{e_0^i} \cap \cdots \cap q_{r-1}^{e_{r-1}^i}\right) = (h_g(q_0))^{e_0^i} \cap \cdots \cap (h_g(q_{r-1}))^{e_{r-1}^i}$ for every $i < m$. Hence

$$(h_g(q_0))^{e_0^i} \cap \cdots \cap (h_g(q_{r-1}))^{e_{r-1}^i} = \{u_i, v_i\} \; (i < m). \tag{4.2.2}$$

So we define $\bar{p}_0, \ldots, \bar{p}_{r-1} \in \mathcal{P}(U)$ by $\bar{p}_k = h_g(q_k)$ for $k < r$.

Let $i < m$ and $d \in D^m$ be fixed. Our goal is to prove that $\eta_{\lambda P,i}(d) \neq \mathbf{0}$.

Define an auxiliary relation $R_{i,d}^{aux} \subseteq U \times U$ by

$$R_{i,d}^{aux} = \{\langle u_i, v_i \rangle \mid d_i = 1\} \cup \{\langle u_i, u_j \rangle, \langle u_i, v_j \rangle \mid d_j = 1, j < m, j \neq i\}. \tag{4.2.3}$$

Note that $u_i \notin range(R_{i,d}^{aux})$.

Define a marked directed graph $\mathcal{F}_{i,d} = (W_{i,d}, R_{i,d}, E_{i,d})$ by

$$\begin{aligned} W_{i,d} &= U, \\ R_{i,d} &= R_{i,d}^{aux} \cup \left(range(R_{i,d}^{aux}) \times range(R_{i,d}^{aux})\right), \\ E_{i,d} &= range(R_{i,d}^{aux}). \end{aligned} \tag{4.2.4}$$

So $u_i \notin E_{i,d}$, $\langle u_i, u_i \rangle \notin R_{i,d}$, and for all $u, v \in W_{i,d}$, $uR_{i,d}v$ implies $v \in range(R_{i,d}^{aux})$.

To be proved that $\mathcal{F}_{i,d}$ is a bounded graph.

- $R_{i,d}$ is transitive. Suppose $u, v, w \in W_{i,d}, uR_{i,d}v$ and $vR_{i,d}w$. Since $vR_{i,d}w$, we have $w \in range(R_{i,d}^{aux})$ (so $u_iR_{i,d}^{aux}w$). Since $uR_{i,d}v$, either $u = u_i$ or $u \in range(R_{i,d}^{aux})$. If $u = u_i$, then $uR_{i,d}w$ (since $u_iR_{i,d}^{aux}w$). If $u \in range(R_{i,d}^{aux})$, then $uR_{i,d}w$ (since $w \in range(R_{i,d}^{aux})$). Thus $uR_{i,d}w$ in both cases.

- $R_{i,d}$ is Euclidean. Let $u, v, w \in W_{i,d}, uR_{i,d}v$ and $uR_{i,d}w$. Then $v, w \in range(R_{i,d}^{aux})$. Hence $vR_{i,d}w$.

- $\forall u, v \in W_{i,d}(uR_{i,d}v \to v \in E_{i,d})$. Let $u, v \in W_{i,d}$ and $uR_{i,d}v$. Then $v \in range(R_{i,d}^{aux})$. Hence $v \in E_{i,d}$.

- $\forall u \in W_{i,d}(u \in E_{i,d} \to uR_{i,d}u)$. Let $u \in E_{i,d}$. Then $u \in range(R_{i,d}^{aux})$. Hence $\langle u, u \rangle \in range(R_{i,d}^{aux}) \times range(R_{i,d}^{aux})$. Therefore $uR_{i,d}u$.

Therefore the complex algebra $\mathbf{P}_{\mathcal{F}_{i,d}}$ is an MBA (by Lemma 2.2.11).

Let $Q = (\bar{p}_0, \dots, \bar{p}_{r-1})$.

To be proved that $\eta_{\lambda Q,i}(d) \neq \mathbf{0}$ (here we work in $\mathbf{P}_{\mathcal{F}_{i,d}}$). Consider

$$\begin{aligned}\eta_{\lambda Q,i}(d) &= \bar{p}_0^{e_0^i} \cap \cdots \cap \bar{p}_{r-1}^{e_{r-1}^i} \cap E'_{i,d} \cap \left(\langle R_{i,d}\rangle \left(\bar{p}_0^{e_0^0} \cap \cdots \cap \bar{p}_{r-1}^{e_{r-1}^0}\right)\right)^{d_0} \cap \dots \\ &\cdots \cap \left(\langle R_{i,d}\rangle \left(\bar{p}_0^{e_0^{m-1}} \cap \cdots \cap \bar{p}_{r-1}^{e_{r-1}^{m-1}}\right)\right)^{d_{m-1}} \\ &= \{u_i, v_i\} \cap E'_{i,d} \cap (\langle R_{i,d}\rangle\{u_0, v_0\})^{d_0} \cap \dots \\ &\cdots \cap (\langle R_{i,d}\rangle\{u_{m-1}, v_{m-1}\})^{d_{m-1}} \text{ [see Equation 4.2.2].}\end{aligned}$$

We are going to prove that the whole expression contains u_i. Since $u_i \notin E_{i,d}$, we have $u_i \in E'_{i,d}$. Consider d_j for $j < m$ with $j \neq i$. There are two cases:

- $d_j = 1$. Then $u_iR_{i,d}^{aux}u_j \Rightarrow u_iR_{i,d}u_j \Rightarrow u_i \in (\langle R_{i,d}\rangle\{u_j, v_j\}) \Rightarrow u_i \in (\langle R_{i,d}\rangle\{u_j, v_j\})^{d_j}$.

- $d_j = -1$. Then $\langle u_i, u_j\rangle \notin R^{aux}_{i,d}$ and $\langle u_i, v_j\rangle \notin R^{aux}_{i,d}$. Hence $\langle u_i, u_j\rangle \notin R_{i,d}$ and $\langle u_i, v_j\rangle \notin R_{i,d}$. Therefore $u_i \notin (\langle R_{i,d}\rangle\{u_j, v_j\})$. So $u_i \in (\langle R_{i,d}\rangle\{u_j, v_j\})^{d_j}$.

So $u_i \in (\langle R_{i,d}\rangle\{u_j, v_j\})^{d_j}$. It remains to prove that $u_i \in (\langle R_{i,d}\rangle\{u_i, v_i\})^{d_i}$. There are two cases:

- $d_i = 1$. Then $u_i R^{aux}_{i,d} v_i \Rightarrow u_i R_{i,d} v_i \Rightarrow u_i \in (\langle R_{i,d}\rangle\{u_i, v_i\}) \Rightarrow u_i \in (\langle R_{i,d}\rangle\{u_i, v_i\})^{d_i}$.

- $d_i = -1$. Then $\langle u_i, v_i\rangle \notin R^{aux}_{i,d}$. Hence $\langle u_i, v_i\rangle \notin R_{i,d}$. Therefore $u_i \notin (\langle R_{i,d}\rangle\{u_i, v_i\})$. So $u_i \in (\langle R_{i,d}\rangle\{u_i, v_i\})^{d_i}$.

Thus $u_i \in (\langle R_{i,d}\rangle\{u_i, v_i\})^{d_i}$ in both cases.

So $\eta_{\lambda Q,i}(d) \neq \mathbf{0}$.

Now define a mapping $f_0 : G \to \{\bar{p}_0, \ldots, \bar{p}_{r-1}\}$ by $f_0(p_k) = \bar{p}_k$ for every $k < r$. Then there is an MBA-homomorphism $f : \mathbf{M} \to \mathbf{P}_{\mathcal{F}_{i,d}}$ which extends f_0 (since $\mathbf{M}$ is free on the set G of generators). Therefore $\mu_{\lambda P,i}(d) \neq \mathbf{0}$ otherwise $\mu_{\lambda Q,i}(d) = f(\mu_{\lambda P,i}(d)) = f(\mathbf{0}) = \mathbf{0}$. □

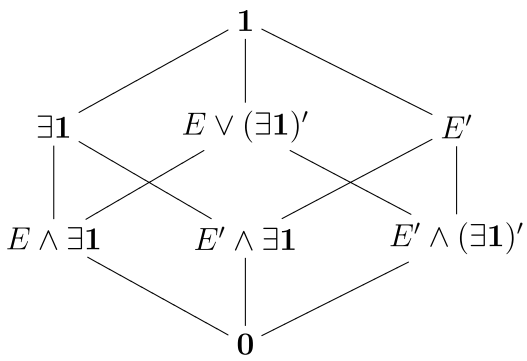
Theorem 4.2.3. *Every MBA freely generated by $r < \omega$ many elements has exactly $3 \cdot 2^r \cdot 2^{2^r - 1}$ atoms.*

Proof. Follows from Lemma 4.2.1 and Lemma 4.2.2. □

Corollary 4.2.4. *Every MBA freely generated by $r < \omega$ many elements has exactly $2^{3 \cdot 2^r \cdot 2^{2^r - 1}}$ elements.*

Let us compare the theorem and its corollary with the monadic case. In [1, Theorem 4] H. Bass proves that the monadic algebra free on r elements has exactly $2^r \cdot 2^{2^r - 1}$ atoms and $2^{2^r \cdot 2^{2^r - 1}}$ elements. So, in particular, the MBA and the monadic algebra freely generated by the empty set (i.e. $r = 0$) have 8 and 2 elements, respectively, and the MBA and the monadic algebra freely generated by one element (i.e. $r = 1$) have $2^{12} = 4096$ and $2^4 = 16$ elements, respectively.

We are going to draw the diagram of the MBA freely generated by the empty set. The elements $E \wedge \exists \mathbf{1}$, $E' \wedge \exists \mathbf{1}$ and $E' \wedge (\exists \mathbf{1})'$ (see μ-, η-expressions on p. 86) are the atoms of the MBA freely generated by the empty set. So the diagram looks as follows:



4.3 Extensions of MBA-homomorphisms of finite MBA's

In this section we give a necessary and sufficient condition under which certain maps between finitely generated MBA's can be extended to MBA-homomorphisms. The section may be considered as a preliminary to the next section.

The following result is known from [1, p. 261]. Suppose $\mathbf{A}$ and $\mathbf{B}$ are Boolean algebras, $\mathbf{A}$ is finite, with $\sigma = \{p_0, \ldots, p_{n-1}\}$ and $\tau = \{t_0, \ldots, t_{n-1}\}$ partitions of $\mathbf{1}$ in $\mathbf{A}$ and $\mathbf{B}$, respectively. Define a map $f_0 : \sigma \to \tau$ by $f_0(p_i) = t_i$ for every $i < n$.

Lemma 4.3.1 (H. Bass). *Suppose σ contains all the atoms of the Boolean algebra $\mathbf{A}$. Then the map f_0 can be extended to a Boolean homomorphism $f : \mathbf{A} \to \mathbf{B}$ iff $t_i = \mathbf{0}$ whenever $p_i = \mathbf{0}$.*

Proof. The necessity is an immediate consequence of the fact that $f(\mathbf{0}) = \mathbf{0}$.

Suppose, conversely, that $t_i = \mathbf{0}$ whenever $p_i = \mathbf{0}$.

To be proved that if $p_i \in \sigma$ is not an atom of $\mathbf{A}$, then $p_i = \mathbf{0}$. Assume $p_i \in \sigma$ is not an atom and $p_i \neq \mathbf{0}$. Then there is at least one atom $a \in \mathbf{A}$ such that $a \leq p_i$. Hence $a = p_j$ for some $j < n$ with $j \neq i$. Therefore $p_i \wedge p_j = p_i \wedge a = a \neq \mathbf{0}$ and $i \neq j$. But σ is a partition.

Define $f : \mathbf{A} \to \mathbf{B}$ by

$$f(p_{i_0} \vee \cdots \vee p_{i_k}) = t_{i_0} \vee \cdots \vee t_{i_k}.$$

This f is well defined since every element of $\mathbf{A}$ is uniquely a supremum of atoms, and if extra p_i's which all equal zero are thrown in on the left, the corresponding t_i's, by hypothesis, contribute nothing new on the right. Moreover, f clearly extends f_0 and commutes with all suprema. That f commutes with complementation follows from the fact that the complement of a supremum of elements in a partition of $\mathbf{1}$ is just the supremum of the remaining elements in that partition. Therefore f is the desired homomorphism. □

Now let $(\mathbf{A}, E, \exists)$ and $(\mathbf{B}, E, \exists)$ be MBA's generated by $\sigma = \{p_0, \ldots, p_{r-1}\} \subseteq \mathbf{A}$ and $\tau = \{t_0, \ldots, t_{r-1}\} \subseteq \mathbf{B}$, respectively. Then let $m = 2^r$, $D^r = \{e^0, \ldots, e^{m-1}\}$, $P = (p_0, \ldots, p_{r-1})$, $T = (t_0, \ldots, t_{r-1})$, and

$$\Omega_{\mathbf{A}} = \{\mu_{\lambda P,i}(d) \mid i < m, d \in D_i^m\} \cup \{\eta_{\lambda P,i}(d) \mid i < m, d \in D^m\}, \tag{4.3.1}$$

$$\Omega_{\mathbf{B}} = \{\mu_{\lambda T,i}(d) \mid i < m, d \in D_i^m\} \cup \{\eta_{\lambda T,i}(d) \mid i < m, d \in D^m\}. \tag{4.3.2}$$

Theorem 4.3.2. *The map $f_0 : \sigma \to \tau$ defined by $f_0(p_j) = t_j$ ($j < r$) can be extended to an MBA-homomorphism $f : \mathbf{A} \to \mathbf{B}$ iff the following two conditions hold:*

1. *For every $i < m$ and $d \in D_i^m$, $\mu_{\lambda T,i}(d) = \mathbf{0}$ whenever $\mu_{\lambda P,i}(d) = \mathbf{0}$,*

2. *For every $i < m$ and $d \in D^m$, $\eta_{\lambda T,i}(d) = \mathbf{0}$ whenever $\eta_{\lambda P,i}(d) = \mathbf{0}$.*

Proof. Part $\Rightarrow$. Suppose f_0 can be extended to an MBA-homomorphism $f : \mathbf{A} \to \mathbf{B}$. Assume $\mu_{\lambda P,i}(d) = \mathbf{0}$ for some $d \in D_i^m$ and $i < m$. Then

$$\mu_{\lambda T,i}(d) = f(\mu_{\lambda P,i}(d)) = f(\mathbf{0}) = \mathbf{0}.$$

Analogously with the second item.

Part $\Leftarrow$. Suppose both (1) and (2) hold. Define a mapping $g_0 : \Omega_{\mathbf{A}} \to \Omega_{\mathbf{B}}$ by

$$g_0(p) = \begin{cases} \mu_{\lambda T,i}(d), & \text{if } p = \mu_{\lambda P,i}(d) \text{ for some } i < m \text{ and } d \in D_i^m \\ \eta_{\lambda T,i}(d), & \text{if } p = \eta_{\lambda P,i}(d) \text{ for some } i < m \text{ and } d \in D_i. \end{cases} \tag{4.3.3}$$

Recall that $\Omega_{\mathbf{A}}$ and $\Omega_{\mathbf{B}}$ are partitions of $\mathbf{1}$ in $\mathbf{A}$ and $\mathbf{B}$, respectively, and $\Omega_{\mathbf{A}}$ contains all the atoms of $\mathbf{A}$. Therefore we can apply Lemma 4.3.1 to g_0. So g_0 can be extended to a Boolean homomorphism $f : \mathbf{A} \to \mathbf{B}$.

To be proved that $f(E) = E$. Consider

$$\begin{aligned} f(E) &= f\left(\bigvee_{i<m, d \in D_i^m} \mu_{\lambda P,i}(d)\right) \text{ [by Lemma 4.1.31]} \\ &= \bigvee_{i<m, d \in D_i^m} f\left(\mu_{\lambda P,i}(d)\right) \text{ [since } f \text{ is a Boolean homomorphism]} \\ &= \bigvee_{i<m, d \in D_i^m} g_0\left(\mu_{\lambda P,i}(d)\right) \text{ [since } f \text{ extends } g_0\text{]} \\ &= \bigvee_{i<m, d \in D_i^m} \mu_{\lambda T,i}(d) \text{ [by definition of } g_0\text{]} \\ &= E \text{ [by Lemma 4.1.31].} \end{aligned}$$

Next to be proved that f preserves $\exists$. It suffices by Definition 2.2.1(5) to prove that f commutes with $\exists$ on atoms of $\mathbf{A}$.

Firstly, let $i < m$ and $d \in D_i^m$. Then

$$
\begin{aligned}
f(\exists\mu_{\lambda P,i}(d)) &= f\left(\bigvee_{\{j|d_j=1\},k<m}\{\mu_{\lambda P,j}(d),\eta_{\lambda P,k}(d)\}\right) \text{ [by Corollary 4.1.29]}\\
&= \bigvee_{\{j|d_j=1\},k<m}\{f(\mu_{\lambda P,j}(d)),f(\eta_{\lambda P,k}(d))\}\\
&\quad\text{[since } f \text{ is a Boolean homomorphism]}\\
&= \bigvee_{\{j|d_j=1\},k<m}\{g_0(\mu_{\lambda P,j}(d)),g_0(\eta_{\lambda P,k}(d))\} \text{ [since } f \text{ extends } g_0]\\
&= \bigvee_{\{j|d_j=1\},k<m}\{\mu_{\lambda T,j}(d),\eta_{\lambda T,k}(d)\} \text{ [by definition of } g_0]\\
&= \exists\mu_{\lambda T,i}(d) \text{ [by Corollary 4.1.29]}\\
&= \exists(g_0(\mu_{\lambda P,i}(d))) \text{ [by definition of } g_0]\\
&= \exists(f(\mu_{\lambda P,i}(d))) \text{ [since } f \text{ extends } g_0]\ .
\end{aligned}
$$

Thus $f(\exists\mu_{\lambda P,i}(d)) = \exists f(\mu_{\lambda P,i}(d))$ for every $i < m$ and $d \in D_i^m$.

Secondly, let $i < m$ and $d \in D^m$. Then $f(\exists(\eta_{\lambda P,i}(d))) = f(\mathbf{0})$ [by Lemma 4.1.18] $= \mathbf{0} = \exists(\eta_{\lambda T,i}(d))$ [by Lemma 4.1.18] $= \exists(g_0(\eta_{\lambda P,i}(d)))$ [by definition of g_0] $= \exists(f(\eta_{\lambda P,i}(d)))$ [since f extends g_0]. Thus $f(\exists(\eta_{\lambda P,i}(d))) = \exists(f(\eta_{\lambda P,i}(d)))$ for every $i < m$ and $d \in D^m$.

So f commutes with $\exists$ on atoms of $\mathbf{A}$. Hence f preserves $\exists$.

It remains only to show that f extends f_0 (i.e. $f(p_j) = t_j$ for every $j < r$). We know that, for every $j < r$,

$$
p_j = \bigvee_{e\in D_j^r}\lambda_P(e) = \bigvee_{e\in D_j^r}\left(p_0^{e_0}\wedge\cdots\wedge p_{j-1}^{e_{j-1}}\wedge p_j^1\wedge p_{j+1}^{e_{j+1}}\wedge\cdots\wedge p_{r-1}^{e_{r-1}}\right) \tag{4.3.4}
$$

and

$$
t_j = \bigvee_{e\in D_j^r}\lambda_T(e) = \bigvee_{e\in D_j^r}\left(t_0^{e_0}\wedge\cdots\wedge t_{j-1}^{e_{j-1}}\wedge t_j^1\wedge t_{j+1}^{e_{j+1}}\wedge\cdots\wedge t_{r-1}^{e_{r-1}}\right). \tag{4.3.5}
$$

It therefore suffices to prove that $f(\lambda_P(e^k)) = \lambda_T(e^k)$ for every $k < m$. For every $k < m$, we have

$$\begin{aligned} f(\lambda_P(e^k)) &= f\left(\bigvee_{d\in D_k^m,\, c\in D^m} \{\mu_{\lambda P,k}(d), \eta_{\lambda P,k}(c)\}\right) \text{ [by Lemma 4.1.30]} \\ &= \bigvee_{d\in D_k^m,\, c\in D^m} \{f(\mu_{\lambda P,k}(d)), f(\eta_{\lambda P,k}(c))\} \\ &= \bigvee_{d\in D_k^m,\, c\in D^m} \{g_0(\mu_{\lambda P,k}(d)), g_0(\eta_{\lambda P,k}(c))\} \\ &= \bigvee_{d\in D_k^m,\, c\in D^m} \{\mu_{\lambda T,k}(d), \eta_{\lambda T,k}(c)\} \\ &= \lambda_T(e^k) \text{ [by Lemma 4.1.30].} \end{aligned}$$

So $f(p_j) = f\left(\bigvee_{e\in D_j^r} \lambda_P(e)\right) = \bigvee_{e\in D_j^r} f(\lambda_P(e)) = \bigvee_{e\in D_j^r} \lambda_T(e) = t_j$ (for every $j < r$). Hence f extends f_0.

Thus f is an MBA-homomorphism which extends f_0. □

4.4 Free MBA's on finite sets themselves

The purpose of this section is to construct free MBA's on finite sets. Firstly, a marked directed graph is defined. Secondly, we prove that this marked directed graph is a bounded graph. Thirdly, certain subsets of the bounded graph are defined. Finally, using results from the previous section, we prove that the complex algebra of the bounded graph is a free MBA on the set consisting of the subsets. For better understanding, two particular cases are considered explicitly.

Let $r < \omega$ be fixed. Our goal is to construct the free MBA on r elements.

Let $m = 2^r$, $x_0, \dots, x_{r-1}$ be variables, and $X = (x_0, \dots, x_{r-1})$. Choose a definite enumeration $D^r = \{e^0, \dots, e^{m-1}\}$ and, as usual, put

$$\Omega = \{\mu_{\lambda X,i}(d) \mid i < m, d \in D_i^m\} \cup \{\eta_{\lambda X,i}(d) \mid i < m, d \in D^m\}.$$

Note that $\mu_{\lambda X,i}(d)$ and $\eta_{\lambda X,i}(d)$ are terms of the type $\{\wedge, \vee, ', \mathbf{0}, \mathbf{1}, E, \exists\}$

over $\{x_0, \ldots, x_{r-1}\}$. Since different terms as such are distinct objects, there are $3 \cdot 2^r \cdot 2^{2^r-1}$ distinct elements in Ω.

Define a marked directed graph $\mathcal{F} = (W, R, E)$ by

- $W = \Omega$,
- $E = \{\mu_{\lambda X,i}(d) \mid i < m, d \in D_i^m\}$,
- $R = \bigcup_{i<m, d\in D_i^m} R[\mu_{\lambda X,i}(d)]$, where

$$R[\mu_{\lambda X,i}(d)] = \{\langle \mu_{\lambda X,j}(d), \mu_{\lambda X,i}(d)\rangle, \langle \eta_{\lambda X,k}(d), \mu_{\lambda X,i}(d)\rangle \mid j < m \text{ such that } d_j = 1, k < m\}. \tag{4.4.1}$$

So $W - E = \{\eta_{\lambda X,i}(d) \mid i < m, d \in D^m\}$. Moreover, $\langle \mu_{\lambda X,i}(d), \mu_{\lambda X,i}(d)\rangle \in R[\mu_{\lambda X,i}(d)]$ (and hence $\langle \mu_{\lambda X,i}(d), \mu_{\lambda X,i}(d)\rangle \in R$) for every $i < m$ and $d \in D_i^m$.

We are going to look at particular cases.

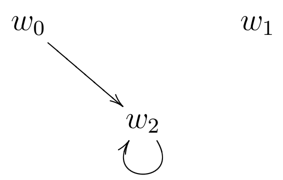
Let $r = 0$. Then, as on page 85, we have

$$\begin{aligned} W &= \{\mu_{\lambda X,i}(d) \mid i < 1, d \in D_i^1\} \cup \{\eta_{\lambda X,i}(d) \mid i < 1, d \in D^1\} \\ &= \{\mu_{\lambda X,i}(d) \mid i = 0, d \in D_0^1 = \{(1)\}\} \\ &\cup \{\eta_{\lambda X,i}(d) \mid i = 0, d \in D^1 = \{(1), (-1)\}\} \\ &= \{\mu_{\lambda X,0}((1))\} \cup \{\eta_{\lambda X,0}((1)), \eta_{\lambda X,0}((-1))\} \\ &= \{E \wedge \exists \mathbf{1}\} \cup \{E' \wedge \exists \mathbf{1}, E' \wedge (\exists \mathbf{1})'\}. \end{aligned}$$

So $E = \{\mu_{\lambda X,0}((1))\} = \{E \wedge \exists \mathbf{1}\}$ and

$$\begin{aligned} R &= \bigcup_{i<1, d\in D_i^1} R[\mu_{\lambda X,i}(d)] = R[\mu_{\lambda X,0}((1))] \\ &= \{\langle \mu_{\lambda X,j}((1)), \mu_{\lambda X,0}((1))\rangle, \langle \eta_{\lambda X,k}((1)), \mu_{\lambda X,0}((1))\rangle \mid \\ &\quad j < 1 \text{ such that } (1)_j = 1, k < 1\} \\ &= \{\langle \mu_{\lambda X,0}((1)), \mu_{\lambda X,0}((1))\rangle, \langle \eta_{\lambda X,0}((1)), \mu_{\lambda X,0}((1))\rangle\} \\ &= \{\langle E \wedge \exists \mathbf{1}, E \wedge \exists \mathbf{1}\rangle, \langle E' \wedge \exists \mathbf{1}, E \wedge \exists \mathbf{1}\rangle\}. \end{aligned}$$

In a picture the marked directed graph looks like



where

$w_0 = \eta_{\lambda X,0}((1)) = E' \wedge \exists \mathbf{1}$,

$w_1 = \eta_{\lambda X,0}((-1)) = E' \wedge (\exists \mathbf{1})'$,

$w_2 = \mu_{\lambda X,0}((1)) = E \wedge \exists \mathbf{1}$.

(This is actually a bounded graph whose complex algebra is the MBA freely generated by the empty set.)

Let $r = 1$. Then, as on page 86, we have

$$
\begin{aligned}
W &= \{\mu_{\lambda X,i}(d) \mid i < 2, d \in D_i^2\} \cup \{\eta_{\lambda X,i}(d) \mid i < 2, d \in D^2\} \\
&= \{\mu_{\lambda X,i}(d) \mid i = 0, d \in D_0^2\} \cup \{\mu_{\lambda X,i}(d) \mid i = 1, d \in D_1^2\} \\
&\cup \{\eta_{\lambda X,i}(d) \mid i = 0, d \in D^2\} \cup \{\eta_{\lambda X,i}(d) \mid i = 1, d \in D^2\} \\
&= \{\mu_{\lambda X,0}((1,1)), \mu_{\lambda X,0}((1,-1))\} \cup \{\mu_{\lambda X,1}((1,1)), \mu_{\lambda X,1}((-1,1))\} \\
&\cup \{\eta_{\lambda X,0}((1,1)), \eta_{\lambda X,0}((-1,1)), \eta_{\lambda X,0}((1,-1)), \eta_{\lambda X,0}((-1,-1))\} \\
&\cup \{\eta_{\lambda X,1}((1,1)), \eta_{\lambda X,1}((-1,1)), \eta_{\lambda X,1}((1,-1)), \eta_{\lambda X,1}((-1,-1))\} \\
&= \{x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0 \wedge E \wedge \exists x_0 \wedge (\exists(x_0'))'\} \\
&\cup \{x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0' \wedge E \wedge (\exists x_0)' \wedge \exists(x_0')\} \\
&\cup \{x_0 \wedge E' \wedge \exists x_0 \wedge \exists(x_0'), x_0 \wedge E' \wedge (\exists x_0)' \wedge \exists(x_0'), \\
&\quad x_0 \wedge E' \wedge \exists x_0 \wedge (\exists(x_0'))', x_0 \wedge E' \wedge (\exists x_0)' \wedge (\exists(x_0'))'\} \\
&\cup \{x_0' \wedge E' \wedge \exists x_0 \wedge \exists(x_0'), x_0' \wedge E' \wedge (\exists x_0)' \wedge \exists(x_0'), \\
&\quad x_0' \wedge E' \wedge \exists x_0 \wedge (\exists(x_0'))', x_0' \wedge E' \wedge (\exists x_0)' \wedge (\exists(x_0'))'\}.
\end{aligned}
$$

So

$$
\begin{aligned}
E &= \{\mu_{\lambda X,0}((1,1)), \mu_{\lambda X,0}((1,-1)), \mu_{\lambda X,1}((1,1)), \mu_{\lambda X,1}((-1,1))\} \\
&= \{x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0 \wedge E \wedge \exists x_0 \wedge (\exists(x_0'))', \\
&\quad x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0' \wedge E \wedge (\exists x_0)' \wedge \exists(x_0')\}
\end{aligned}
$$

and

$$R = \bigcup_{i<2, d\in D_i^2} R[\mu_{\lambda X,i}(d)] = \bigcup_{d\in D_0^2} R[\mu_{\lambda X,0}(d)] \cup \bigcup_{d\in D_1^2} R[\mu_{\lambda X,1}(d)]$$
$$= R[\mu_{\lambda X,0}((1,1))] \cup R[\mu_{\lambda X,0}((1,-1))] \cup R[\mu_{\lambda X,1}((1,1))] \cup R[\mu_{\lambda X,1}((-1,1))],$$

where

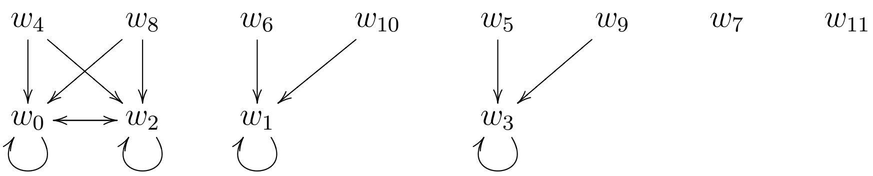
$$R[\mu_{\lambda X,0}((1,1))] = \{\langle \mu_{\lambda X,j}((1,1)), \mu_{\lambda X,0}((1,1))\rangle, \langle \eta_{\lambda X,k}((1,1)), \mu_{\lambda X,0}((1,1))\rangle \mid$$
$$j < 2 \text{ such that } (1,1)_j = 1, k < 2\}$$
$$= \{\langle \mu_{\lambda X,0}((1,1)), \mu_{\lambda X,0}((1,1))\rangle, \langle \mu_{\lambda X,1}((1,1)), \mu_{\lambda X,0}((1,1))\rangle,$$
$$\langle \eta_{\lambda X,0}((1,1)), \mu_{\lambda X,0}((1,1))\rangle, \langle \eta_{\lambda X,1}((1,1)), \mu_{\lambda X,0}((1,1))\rangle\}$$
$$= \{\langle x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle,$$
$$\langle x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle,$$
$$\langle x_0 \wedge E' \wedge \exists x_0 \wedge \exists(x_0'), x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle,$$
$$\langle x_0' \wedge E' \wedge \exists x_0 \wedge \exists(x_0'), x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle\},$$

$$R[\mu_{\lambda X,0}((1,-1))] = \{\langle \mu_{\lambda X,j}((1,-1)), \mu_{\lambda X,0}((1,-1))\rangle,$$
$$\langle \eta_{\lambda X,k}((1,-1)), \mu_{\lambda X,0}((1,-1))\rangle \mid j < 2 \text{ such that } (1,-1)_j = 1, k < 2\}$$
$$= \{\langle \mu_{\lambda X,0}((1,-1)), \mu_{\lambda X,0}((1,-1))\rangle, \langle \eta_{\lambda X,0}((1,-1)), \mu_{\lambda X,0}((1,-1))\rangle,$$
$$\langle \eta_{\lambda X,1}((1,-1)), \mu_{\lambda X,0}((1,-1))\rangle\}$$
$$= \{\langle x_0 \wedge E \wedge \exists x_0 \wedge (\exists(x_0'))', x_0 \wedge E \wedge \exists x_0 \wedge (\exists(x_0'))'\rangle,$$
$$\langle x_0 \wedge E' \wedge \exists x_0 \wedge (\exists(x_0'))', x_0 \wedge E \wedge \exists x_0 \wedge (\exists(x_0'))'\rangle,$$
$$\langle x_0' \wedge E' \wedge \exists x_0 \wedge (\exists(x_0'))', x_0 \wedge E \wedge \exists x_0 \wedge (\exists(x_0'))'\rangle\},$$

$R[\mu_{\lambda X,1}((1,1))] = \{\langle \mu_{\lambda X,j}((1,1)), \mu_{\lambda X,1}((1,1))\rangle, \langle \eta_{\lambda X,k}((1,1)), \mu_{\lambda X,1}((1,1))\rangle \mid$
$j < 2$ such that $(1,1)_j = 1, k < 2\}$
$= \{\langle \mu_{\lambda X,0}((1,1)), \mu_{\lambda X,1}((1,1))\rangle, \langle \mu_{\lambda X,1}((1,1)), \mu_{\lambda X,1}((1,1))\rangle,$
$\langle \eta_{\lambda X,0}((1,1)), \mu_{\lambda X,1}((1,1))\rangle, \langle \eta_{\lambda X,1}((1,1)), \mu_{\lambda X,1}((1,1))\rangle\}$
$= \{\langle x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle,$
$\langle x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0'), x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle,$
$\langle x_0 \wedge E' \wedge \exists x_0 \wedge \exists(x_0'), x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle,$
$\langle x_0' \wedge E' \wedge \exists x_0 \wedge \exists(x_0'), x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0')\rangle\},$

$R[\mu_{\lambda X,1}((-1,1))] = \{\langle \mu_{\lambda X,j}((-1,1)), \mu_{\lambda X,1}((-1,1))\rangle,$
$\langle \eta_{\lambda X,k}((-1,1)), \mu_{\lambda X,1}((-1,1))\rangle \mid j < 2$ such that $(-1,1)_j = 1, k < 2\}$
$= \{\langle \mu_{\lambda X,1}((-1,1)), \mu_{\lambda X,1}((-1,1))\rangle, \langle \eta_{\lambda X,0}((-1,1)), \mu_{\lambda X,1}((-1,1))\rangle,$
$\langle \eta_{\lambda X,1}((-1,1)), \mu_{\lambda X,1}((-1,1))\rangle\}$
$= \{\langle x_0' \wedge E \wedge (\exists x_0)' \wedge \exists(x_0'), x_0' \wedge E \wedge (\exists x_0)' \wedge \exists(x_0')\rangle,$
$\langle x_0 \wedge E' \wedge (\exists x_0)' \wedge \exists(x_0'), x_0' \wedge E \wedge (\exists x_0)' \wedge \exists(x_0')\rangle,$
$\langle x_0' \wedge E' \wedge (\exists x_0)' \wedge \exists(x_0'), x_0' \wedge E \wedge (\exists x_0)' \wedge \exists(x_0')\rangle\}.$

In a picture the marked directed graph looks like



where

$w_0 = \mu_{\lambda X,0}((1,1)) = x_0 \wedge E \wedge \exists x_0 \wedge \exists(x_0'),$
$w_1 = \mu_{\lambda X,0}((1,-1)) = x_0 \wedge E \wedge \exists x_0 \wedge (\exists(x_0'))',$
$w_2 = \mu_{\lambda X,1}((1,1)) = x_0' \wedge E \wedge \exists x_0 \wedge \exists(x_0'),$
$w_3 = \mu_{\lambda X,1}((-1,1)) = x_0' \wedge E \wedge (\exists x_0)' \wedge \exists(x_0'),$
$w_4 = \eta_{\lambda X,0}((1,1)) = x_0 \wedge E' \wedge \exists x_0 \wedge \exists(x_0'),$
$w_5 = \eta_{\lambda X,0}((-1,1)) = x_0 \wedge E' \wedge (\exists x_0)' \wedge \exists(x_0'),$
$w_6 = \eta_{\lambda X,0}((1,-1)) = x_0 \wedge E' \wedge \exists x_0 \wedge (\exists(x_0'))',$

$w_7 = \eta_{\lambda X,0}((-1,-1)) = x_0 \wedge E' \wedge (\exists x_0)' \wedge (\exists(x_0'))'$,
$w_8 = \eta_{\lambda X,1}((1,1)) = x_0' \wedge E' \wedge \exists x_0 \wedge \exists(x_0')$,
$w_9 = \eta_{\lambda X,1}((-1,1)) = x_0' \wedge E' \wedge (\exists x_0)' \wedge \exists(x_0')$,
$w_{10} = \eta_{\lambda X,1}((1,-1)) = x_0' \wedge E' \wedge \exists x_0 \wedge (\exists(x_0'))'$,
$w_{11} = \eta_{\lambda X,1}((-1,-1)) = x_0' \wedge E' \wedge (\exists x_0)' \wedge (\exists(x_0'))'$.

(This is actually a bounded graph whose complex algebra is an MBA freely generated by a certain subset of $\{w_0, \ldots, w_{11}\}$.)

We are going to prove several useful lemmas.

Lemma 4.4.1. *Suppose $i, j < m$.*

1. *For $d \in D_i^m$ and $c \in D_j^m$, $\mu_{\lambda X,i}(d) = \mu_{\lambda X,j}(c)$ iff both $i = j$ and $d = c$.*
2. *For $d, c \in D^m$, $\eta_{\lambda X,i}(d) = \eta_{\lambda X,j}(c)$ iff both $i = j$ and $d = c$.*
3. *For $d \in D_i^m$ and $c \in D^m$, $\mu_{\lambda X,i}(d) \neq \eta_{\lambda X,j}(c)$.*

Proof. Obvious. □

Lemma 4.4.2. *Suppose $u, v \in W$.*

1. *If $\langle u, v\rangle \in R$, then $v = \mu_{\lambda X,i}(d)$ for some $i < m$ and $d \in D_i^m$.*
2. *For $i < m$ and $d \in D_i^m$, $\langle u, \mu_{\lambda X,i}(d)\rangle \in R$ iff $\langle u, \mu_{\lambda X,i}(d)\rangle \in R[\mu_{\lambda X,i}(d)]$.*
3. *For $i, j < m$, $d \in D_j^m$ and $c \in D_i^m$, $\langle \mu_{\lambda X,j}(d), \mu_{\lambda X,i}(c)\rangle \in R[\mu_{\lambda X,i}(c)]$ iff $d = c$.*
4. *For $i, j < m$, $d \in D^m$ and $c \in D_j^m$, $\langle \eta_{\lambda X,i}(d), \mu_{\lambda X,j}(c)\rangle \in R[\mu_{\lambda X,j}(c)]$ iff $d = c$.*

Proof. 1. Let $\langle u, v\rangle \in R$. Then $\langle u, v\rangle \in R[\mu_{\lambda X,i}(d)]$ for some $i < m$ and $d \in D_i^m$. Hence $v = \mu_{\lambda X,i}(d)$ (by definition of $R[\mu_{\lambda X,i}(d)]$).

2. Let $i < m$ and $d \in D_i^m$. The $\Leftarrow$ part is by definition of R. Part $\Rightarrow$. Since $\langle u, \mu_{\lambda X,i}(d)\rangle \in R$, we have $\langle u, \mu_{\lambda X,i}(d)\rangle \in R[\mu_{\lambda X,j}(c)]$ for some $j < m$ and $c \in D_j^m$. Hence $\mu_{\lambda X,i}(d) = \mu_{\lambda X,j}(c)$ (by definition of $R[\mu_{\lambda X,j}(c)]$). Thus $\langle u, \mu_{\lambda X,i}(d)\rangle \in R[\mu_{\lambda X,i}(d)]$.

3. Let $d \in D_j^m$ and $c \in D_i^m$. The $\Leftarrow$ part is by definition of $R[\mu_{\lambda X,i}(c)]$. Part $\Rightarrow$. Since $\langle \mu_{\lambda X,j}(d), \mu_{\lambda X,i}(c)\rangle \in R[\mu_{\lambda X,i}(c)]$, we have $\mu_{\lambda X,j}(d) = \mu_{\lambda X,k}(c)$ for some $k < m$ with $c_k = 1$. Hence $d = c$ (and $j = k$).

4. Let $i, j < m$, $d \in D^m$ and $c \in D_j^m$. The $\Leftarrow$ part is by definition of $R[\mu_{\lambda X,j}(c)]$. Part $\Rightarrow$. Since $\langle \eta_{\lambda X,i}(d), \mu_{\lambda X,j}(c)\rangle \in R[\mu_{\lambda X,j}(c)]$, we have $\eta_{\lambda X,i}(d) = \eta_{\lambda X,k}(c)$ for some $k < m$. Hence $d = c$ (and $i = k$). □

Lemma 4.4.3. *R is transitive.*

Proof. Let $u, v, w \in W$, uRv and vRw. To be proved uRw.

Since uRv, we have $\langle u, v\rangle \in R[\mu_{\lambda X,i_0}(d)]$ for some $i_0 < m$ and $d \in D_{i_0}^m$. Hence $v = \mu_{\lambda X,i_0}(d)$, and either $u = \mu_{\lambda X,j_0}(d)$, for some $j_0 \in \{j < m \mid d_j = 1\}$, or $u = \eta_{\lambda X,k_0}(d)$ for some $k_0 < m$.

Since vRw, we have $\langle v, w\rangle \in R[\mu_{\lambda X,i_1}(c)]$ for some $i_1 < m$ and $c \in D_{i_1}^m$. Hence $w = \mu_{\lambda X,i_1}(c)$, and either $v = \mu_{\lambda X,j_1}(c)$, for some $j_1 \in \{j < m \mid c_j = 1\}$, or $v = \eta_{\lambda X,k_1}(c)$, for some $k_1 < m$.

Since we already know that $v = \mu_{\lambda X,i_0}(d)$, we obtain that $\mu_{\lambda X,i_0}(d) = \mu_{\lambda X,j_1}(c)$. So $d = c$ (and $i_0 = j_1$). Hence $w = \mu_{\lambda X,i_1}(d)$. There are two cases:

- If $u = \mu_{\lambda X,j_0}(d)$ (for some $j_0 \in \{j < m \mid d_j = 1\}$), then $\langle u, w\rangle = \langle \mu_{\lambda X,j_0}(d), \mu_{\lambda X,i_1}(d)\rangle \in R[\mu_{\lambda X,i_1}(d)]$.
- If $u = \eta_{\lambda X,k_0}(d)$ (for some $k_0 < m$), then $\langle u, w\rangle = \langle \eta_{\lambda X,k_0}(d), \mu_{\lambda X,i_1}(d)\rangle \in R[\mu_{\lambda X,i_1}(d)]$.

Thus in both cases $\langle u, w\rangle \in R[\mu_{\lambda X,i_1}(d)]$. So $\langle u, w\rangle \in R$. □

Lemma 4.4.4. *R is Euclidean.*

Proof. Let $u, v, w \in W$, uRv and uRw. To be proved vRw.

Since uRv, we have $\langle u, v\rangle \in R[\mu_{\lambda X,i_0}(d)]$ for some $i_0 < m$ and $d \in D_{i_0}^m$. Hence $v = \mu_{\lambda X,i_0}(d)$, and either $u = \mu_{\lambda X,j_0}(d)$, for some $j_0 \in \{j < m \mid d_j = 1\}$, or $u = \eta_{\lambda X,k_0}(d)$, for some $k_0 < m$.

Since uRw, we have $\langle u, w\rangle \in R[\mu_{\lambda X,i_1}(c)]$ for some $i_1 < m$ and $c \in D_{i_1}^m$. Hence $w = \mu_{\lambda X,i_1}(c)$, and either $u = \mu_{\lambda X,j_1}(c)$, for some $j_1 \in \{j < m \mid c_j = 1\}$, or $u = \eta_{\lambda X,k_1}(c)$, for some $k_1 < m$.

There are actually two cases:

- $u = \mu_{\lambda X, j_0}(d)$ (for some $j_0 \in \{j < m \mid d_j = 1\}$) and $u = \mu_{\lambda X, j_1}(c)$ (for some $j_1 \in \{j < m \mid c_j = 1\}$).
- $u = \eta_{\lambda X, k_0}(d)$ (for some $k_0 < m$) and $u = \eta_{\lambda X, k_1}(c)$ (for some $k_1 < m$).

From each of them follows that $d = c$.

Thus $d = c$. Hence $v = \mu_{\lambda X, i_0}(c)$. Therefore $\langle v, w\rangle = \langle \mu_{\lambda X, i_0}(c), \mu_{\lambda X, i_1}(c)\rangle \in R[\mu_{\lambda X, i_1}(c)]$. So $\langle v, w\rangle \in R$. □

Lemma 4.4.5. $\forall u, v \in W(uRv \to v \in E)$.

Proof. Let $u, v \in W$ and uRv. Then $\langle u, v\rangle \in R[\mu_{\lambda X, i}(d)]$ for some $i < m$ and $d \in D^m_i$. Hence $v = \mu_{\lambda X, i}(d)$. So $v \in E$. □

Lemma 4.4.6. $\forall u \in W(u \in E \to uRu)$.

Proof. Let $u \in W$ and $u \in E$. Then $u = \mu_{\lambda X, i}(d)$ for some $i < m$ and $d \in D^m_i$. Since $d \in D^m_i$, we get that $i \in \{j < m \mid d_j = 1\}$. Therefore $\langle u, u\rangle \in R[\mu_{\lambda X, i}(d)]$. So $\langle u, u\rangle \in R$. □

Theorem 4.4.7. *The marked directed graph $\mathcal{F} = (W, R, E)$ is a bounded graph and the complex algebra $\mathbf{P}_{\mathcal{F}}$ is an MBA.*

Proof. It follows from Lemmas 4.4.3 - 4.4.6 that $\mathcal{F}$ is a bounded graph. Therefore $\mathbf{P}_{\mathcal{F}}$ is an MBA (by Lemma 2.2.11). □

After the following technical facts, we will define r many elements in $\mathcal{P}(W)$ which freely generate the MBA $\mathbf{P}_{\mathcal{F}}$.

Lemma 4.4.8. *For every $i < m$, $\bigcap_{k<r} (D^r_k)^{e^i_k} = \{e^i\}$ where*

$$(D^r_k)^{e^i_k} = \begin{cases} D^r_k, \text{ if } e^i_k = 1 \\ D^r - D^r_k, \text{ if } e^i_k = -1 \end{cases} \quad \text{(for every } k < r\text{).}$$

Proof. Let $i < m$ be fixed.

Part $\subseteq$, suppose $e \in \bigcap_{k<r} (D^r_k)^{e^i_k}$. Then $e \in (D^r_k)^{e^i_k}$ for all $k < r$. Hence

$$e_k = \begin{cases} 1, \text{ if } e^i_k = 1 \\ -1, \text{ if } e^i_k = -1 \end{cases} \quad \text{(for every } k < r\text{).}$$

So $e_k = e^i_k$ for every $k < r$. Thus $e = e^i$.

Part $\supseteq$. For every $k < r$, there are two cases:

- If $e^i_k = 1$, then $e^i \in D^r_k$ and $(D^r_k)^{e^i_k} = D^r_k$; hence $e^i \in (D^r_k)^{e^i_k}$.
- If $e^i_k = -1$, then $e^i \in D^r - D^r_k$ and $(D^r_k)^{e^i_k} = D^r - D^r_k$; hence $e^i \in (D^r_k)^{e^i_k}$.

Thus in both cases $e^i \in (D^r_k)^{e^i_k}$. So $e^i \in \bigcap_{k<r}(D^r_k)^{e^i_k}$. □

Definition 4.4.9. *For every $k < r$, define $\Delta_k \subseteq m(= \{0, \ldots, m-1\})$ by*

$$\Delta_k = \{j < m \mid e^j \in D^r_k\}. \tag{4.4.2}$$

(Recall that $D^r = \{e^0, \ldots, e^{m-1}\}$.)

Definition 4.4.10. *For every $l \in \{\pm 1\}$ and $k < r$, define*

$$(\Delta_k)^l = \begin{cases} \Delta_k, \text{ if } l = 1 \\ m - \Delta_k, \text{ if } l = -1. \end{cases} \tag{4.4.3}$$

Corollary 4.4.11. *For every $i < m$, $\bigcap_{k<r}(\Delta_k)^{e^i_k} = \{i\}$.*

Proof. Let $i < m$ be fixed. Since $m - \Delta_k = \{j < m \mid e^j \in (D^r - D^r_k)\}$, we can write $(\Delta_k)^{e^i_k} = \{j < m \mid e^j \in (D^r_k)^{e^i_k}\}$. Therefore, by Lemma 4.4.8,

$$\bigcap_{k<r}(\Delta_k)^{e^i_k} = \bigcap_{k<r}\{j < m \mid e^j \in (D^r_k)^{e^i_k}\} = \Big\{ j < m \mid e^j \in \bigcap_{k<r}(D^r_k)^{e^i_k} \Big\} = \{i\}.$$

□

We are now ready to specify r many elements in $\mathcal{P}(W)$ which freely generate the MBA $\mathbf{P}_{\mathcal{F}}$. For every $k < r$, define $p_k \in \mathcal{P}(W)$ by

$$p_k = \bigcup_{j\in\Delta_k} \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D^m_j, c \in D^m\}. \tag{4.4.4}$$

Consider particular cases. If $r = 0$, then we do not have any p_k. Let $r = 1$. Then $k < r$ implies $k = 0$. Hence we have only one $p_0 \in \mathcal{P}(W)$. Since

$$\Delta_0 = \{j < 2 \mid e^j \in D_0^1 = \{e^0\} = \{(1)\}\} = \{0\},$$
$$D_0^2 = \{(1,1),(1,-1)\}$$

and

$$D^2 = \{(1,1),(-1,1),(1,-1),(-1,-1)\},$$

we obtain

$$\begin{aligned} p_0 &= \bigcup_{j\in\Delta_0} \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^2, c \in D^2\} \\ &= \{\mu_{\lambda X,0}(d), \eta_{\lambda X,0}(c) \mid d \in D_0^2, c \in D^2\} \\ &= \{\mu_{\lambda X,0}((1,1)), \mu_{\lambda X,0}((1,-1)), \eta_{\lambda X,0}((1,1)), \eta_{\lambda X,0}((-1,1)), \\ &\eta_{\lambda X,0}((1,-1)), \eta_{\lambda X,0}((-1,-1))\}. \end{aligned}$$

Using notation on page 108 we may write $p_0 = \{w_0, w_1, w_4, w_5, w_6, w_7\}$.

Let $P = (p_0, \dots, p_{r-1})$.

Note that, by Lemma 4.4.1,

$$\Xi = \big\{\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\} \mid j < m\big\} \tag{4.4.5}$$

is a set-theoretic partition of W, i.e. the intersection of two distinct elements in Ξ is the empty set and the union of all elements in Ξ is W.

Lemma 4.4.12. *For every $i < m$,*

$$\lambda_P(e^i) = \{\mu_{\lambda X,i}(d), \eta_{\lambda X,i}(c) \mid d \in D_i^m, c \in D^m\}. \tag{4.4.6}$$

Proof. Let $i < m$ be fixed. Then

$$\begin{aligned} \lambda_P(e^i) &= \bigcap_{k<r} p_k^{e_k^i} = \bigcap_{k<r} \left(\bigcup_{j\in\Delta_k} \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\} \right)^{e_k^i} \\ &= \bigcap_{k<r} \left(\bigcup_{j\in(\Delta_k)^{e_k^i}} \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\} \right) \\ &= \bigcup_{j\in\left(\bigcap_{k<r}(\Delta_k)^{e_k^i}\right)} \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\} \\ &= \{\mu_{\lambda X,i}(d), \eta_{\lambda X,i}(c) \mid d \in D_i^m, c \in D^m\} \text{ [by Corollary 4.4.11].} \end{aligned}$$

□

In the next two lemmas we will prove that $\mu_{\lambda P,i}(d) \neq \mathbf{0}$ (for every $i < m$ and $d \in D_i^m$) and $\eta_{\lambda P,i}(d) \neq \mathbf{0}$ (for every $i < m$ and $d \in D^m$). The proofs are similar.

Lemma 4.4.13. *For every $i_0 < m$ and $d^* \in D_{i_0}^m$, $\mu_{\lambda P,i_0}(d^*) = \{\mu_{\lambda X,i_0}(d^*)\}$. (Note that $\mu_{\lambda P,i_0}(d^*)$ is an MBA-expression and $\{\mu_{\lambda X,i_0}(d^*)\}$ is a one-element subset of W.)*

Proof. Let $i_0 < m$ and $d^* \in D_{i_0}^m$ be fixed. Then

$$\begin{aligned}
\mu_{\lambda P,i_0}(d^*) &= \lambda_P(e^{i_0}) \cap E \cap \bigcap_{j<m} (\langle R\rangle \lambda_P(e^j))^{d_j^*} \\
&= \{\mu_{\lambda X,i_0}(d), \eta_{\lambda X,i_0}(c) \mid d \in D_{i_0}^m, c \in D^m\} \\
&\cap \{\mu_{\lambda X,i}(d) \mid d \in D_i^m, i < m\} \\
&\cap \Big(\bigcap_{j<m} \big(\langle R\rangle \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}\big)^{d_j^*} \Big)
\end{aligned}$$

(by Lemma 4.4.12 and definition of E).

Part $\subseteq$. Suppose $u \in \mu_{\lambda P,i_0}(d^*)(u \in W)$. Then, in particular, $u \in \lambda_P(e^{i_0}) \cap E$. Hence $u = \mu_{\lambda X,i_0}(\bar{d})$ for some $\bar{d} \in D_{i_0}^m$. To be proved that $\bar{d} = d^*$ (and so $u = \mu_{\lambda X,i_0}(d^*)$). For every coordinate d_j^*, $j < m$, there are two cases:

Case 1 $d_j^* = 1$. Then $u \in \langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$ (since $u \in (\langle R\rangle \lambda_P(e^j))^{d_j^*}$). Hence $\langle u, \mu_{\lambda X,j}(d)\rangle \in R$ for some $d \in D_j^m$ (by definition of $\langle R\rangle$ and Lemma 4.4.2(1)). Therefore $\langle u, \mu_{\lambda X,j}(d)\rangle \in R[\mu_{\lambda X,j}(d)]$ (by Lemma 4.4.2(2)). So $\langle \mu_{\lambda X,i_0}(\bar{d}), \mu_{\lambda X,j}(d)\rangle \in R[\mu_{\lambda X,j}(d)]$. Hence $\bar{d} = d$ (by Lemma 4.4.2(3)). Thus $\bar{d}_j = 1$ (since $d \in D_j^m$). Therefore $d_j^* = \bar{d}_j$.

Case 2 $d_j^* = -1$. Then $u \notin \langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$ (since $u \in (\langle R\rangle \lambda_P(e^j))^{d_j^*}$). Hence, in particular, $\langle u, \mu_{\lambda X,j}(d)\rangle \notin R$ for all $d \in D_j^m$ (by definition of $\langle R\rangle$). Therefore $\langle u, \mu_{\lambda X,j}(d)\rangle \notin R[\mu_{\lambda X,j}(d)]$ for all

$d \in D_j^m$ (by Lemma 4.4.2(2)). So $\langle \mu_{\lambda X,i_0}(\bar{d}), \mu_{\lambda X,j}(d)\rangle \notin R[\mu_{\lambda X,j}(d)]$ for all $d \in D_j^m$. We claim that $\bar{d} \neq d$ for all $d \in D_j^m$. If $\bar{d} = d$ for some $d \in D_j^m$, then $\langle \mu_{\lambda X,i_0}(d), \mu_{\lambda X,j}(d)\rangle \notin R[\mu_{\lambda X,j}(d)]$ which contradicts the definition of $R[\mu_{\lambda X,j}(d)]$. Thus $\bar{d}_j = -1$. Therefore $d_j^* = \bar{d}_j$.

So $\bar{d}_j = d_j^*$ for every $j < m$. Thus $\bar{d} = d^*$. Hence $u = \mu_{\lambda X,i_0}(d^*)$.

Part $\supseteq$. It remains to prove that $\mu_{\lambda X,i_0}(d^*) \in \lambda_P(e^{i_0}) \cap E$ and $\mu_{\lambda X,i_0}(d^*) \in (\langle R\rangle \lambda_P(e^j))^{d_j^*}$ for every $j < m$. Since $d^* \in D_{i_0}^m$, we have $\mu_{\lambda X,i_0}(d^*) \in \lambda_P(e^{i_0}) \cap E$. For every $j < m$, there are two cases:

Case 1 $d_j^* = 1$. Then $d^* \in D_j^m$. Hence $\mu_{\lambda X,j}(d^*)$ is defined and $\mu_{\lambda X,j}(d^*) \in \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. Since $d^* \in D_{i_0}^m$, we have $\langle \mu_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d^*)\rangle \in R[\mu_{\lambda X,j}(d^*)]$. Therefore $\langle \mu_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d^*)\rangle \in R$. So $\mu_{\lambda X,i_0}(d^*) \in \langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. Thus $\mu_{\lambda X,i_0}(d^*) \in (\langle R\rangle \lambda_P(e^j))^{d_j^*}$.

Case 2 $d_j^* = -1$. Then $d^* \notin D_j^m$. We claim that $\mu_{\lambda X,i_0}(d^*) \notin \langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. Assume that $\mu_{\lambda X,i_0}(d^*) \in \langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. Then $\langle \mu_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d)\rangle \in R$ for some $d \in D_j^m$ (by definition of $\langle R\rangle$ and Lemma 4.4.2(1)). Hence $\langle \mu_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d)\rangle \in R[\mu_{\lambda X,j}(d)]$ (by Lemma 4.4.2(2)). Therefore $d^* = d$ (by Lemma 4.4.2(3)). So $d^* \in D_j^m$ (since $d \in D_j^m$). But $d^* \notin D_j^m$. Thus $\mu_{\lambda X,i_0}(d^*) \in (\langle R\rangle \lambda_P(e^j))^{d_j^*}$.

So we have proved that $\mu_{\lambda X,i_0}(d^*) \in (\langle R\rangle \lambda_P(e^j))^{d_j^*}$ for every $j < m$. Therefore we conclude that $\mu_{\lambda X,i_0}(d^*) \in \mu_{\lambda P,i_0}(d^*)$. □

Lemma 4.4.14. *For every $i_0 < m$ and $d^* \in D^m$, $\eta_{\lambda P,i_0}(d^*) = \{\eta_{\lambda X,i_0}(d^*)\}$.*

Proof. Let $i_0 < m$ and $d^* \in D^m_{i_0}$ be fixed. Then

$$\begin{aligned}\eta_{\lambda P,i_0}(d^*) &= \lambda_P(e^{i_0}) \cap E' \cap \bigcap_{j<m} (\langle R\rangle \lambda_P(e^j))^{d^*_j} \\ &= \{\mu_{\lambda X,i_0}(d), \eta_{\lambda X,i_0}(c) \mid d \in D^m_{i_0}, c \in D^m\} \\ &\cap \{\eta_{\lambda X,i}(c) \mid c \in D^m, i < m\} \\ &\cap \Big(\bigcap_{j<m} \big(\langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D^m_j, c \in D^m\}\big)^{d^*_j}\Big)\end{aligned}$$

(since $E' = W - E$ and by Lemma 4.4.12).

Part $\subseteq$. Suppose $u \in \eta_{\lambda P,i_0}(d^*)$ ($u \in W$). Then, in particular, $u \in \lambda_P(e^{i_0}) \cap E'$. Hence $u = \eta_{\lambda X,i_0}(\bar{c})$ for some $\bar{c} \in D^m$. To be proved that $\bar{c} = d^*$ (and so $u = \eta_{\lambda X,i_0}(d^*)$). For every coordinate d^*_j, $j < m$, there are two cases:

Case 1 $d^*_j = 1$. Then $u \in \langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D^m_j, c \in D^m\}$ (since $u \in (\langle R\rangle\lambda_P(e^j))^{d^*_j}$). Hence $\langle u, \mu_{\lambda X,j}(d)\rangle \in R$ for some $d \in D^m_j$ (by definition of $\langle R\rangle$ and Lemma 4.4.2(1)). Therefore $\langle u, \mu_{\lambda X,j}(d)\rangle \in R[\mu_{\lambda X,j}(d)]$ (by Lemma 4.4.2(2)). So $\langle \eta_{\lambda X,i_0}(\bar{c}), \mu_{\lambda X,j}(d)\rangle \in R[\mu_{\lambda X,j}(d)]$. Hence $\bar{c} = d$ (by Lemma 4.4.2(4)). Thus $\bar{c}_j = 1$ (since $d \in D^m_j$). Therefore $d^*_j = \bar{c}_j$.

Case 2 $d^*_j = -1$. Then $u \notin \langle R\rangle\{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D^m_j, c \in D^m\}$ (since $u \in (\langle R\rangle\lambda_P(e^j))^{d^*_j}$). Hence, in particular, $\langle u, \mu_{\lambda X,j}(d)\rangle \notin R$ for all $d \in D^m_j$ (by definition of $\langle R\rangle$). Therefore $\langle u, \mu_{\lambda X,j}(d)\rangle \notin R[\mu_{\lambda X,j}(d)]$ for all $d \in D^m_j$ (by Lemma 4.4.2(2)). So $\langle \eta_{\lambda X,i_0}(\bar{c}), \mu_{\lambda X,j}(d)\rangle \notin R[\mu_{\lambda X,j}(d)]$ for all $d \in D^m_j$. We claim that $\bar{c} \neq d$ for all $d \in D^m_j$. If $\bar{c} = d$ for some $d \in D^m_j$, then $\langle \eta_{\lambda X,i_0}(d), \mu_{\lambda X,j}(d)\rangle \notin R[\mu_{\lambda X,j}(d)]$ which contradicts the definition of $R[\mu_{\lambda X,j}(d)]$. Thus $\bar{c}_j = -1$. Therefore $d^*_j = \bar{c}_j$.

So $\bar{c}_j = d^*_j$ for every $j < m$. Thus $\bar{c} = d^*$. Hence $u = \eta_{\lambda X,i_0}(d^*)$.

Part $\supseteq$. It remains to prove that $\eta_{\lambda X,i_0}(d^*) \in \lambda_P(e^{i_0}) \cap E'$ and $\eta_{\lambda X,i_0}(d^*) \in (\langle R\rangle\lambda_P(e^j))^{d^*_j}$ for every $j < m$. It is clear that $\eta_{\lambda X,i_0}(d^*) \in \lambda_P(e^{i_0}) \cap E'$. For every $j < m$, there are two cases:

Case 1 $d_j^* = 1$. Then $d^* \in D_j^m$. Hence $\mu_{\lambda X,j}(d^*)$ is defined and $\mu_{\lambda X,j}(d^*) \in \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. By definition of $R[\mu_{\lambda X,j}(d^*)]$, $\langle \eta_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d^*) \rangle \in R[\mu_{\lambda X,j}(d^*)]$. Therefore $\langle \eta_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d^*) \rangle \in R$. So $\eta_{\lambda X,i_0}(d^*) \in \langle R \rangle \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. Thus $\eta_{\lambda X,i_0}(d^*) \in (\langle R \rangle \lambda_P(e^j))^{d_j^*}$.

Case 2 $d_j^* = -1$. Then $d^* \notin D_j^m$. We claim that $\eta_{\lambda X,i_0}(d^*) \notin \langle R \rangle \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. Assume that $\eta_{\lambda X,i_0}(d^*) \in \langle R \rangle \{\mu_{\lambda X,j}(d), \eta_{\lambda X,j}(c) \mid d \in D_j^m, c \in D^m\}$. Then $\langle \eta_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d) \rangle \in R$ for some $d \in D_j^m$ (by definition of $\langle R \rangle$ and Lemma 4.4.2(1)). Hence $\langle \eta_{\lambda X,i_0}(d^*), \mu_{\lambda X,j}(d) \rangle \in R[\mu_{\lambda X,j}(d)]$ (by Lemma 4.4.2(2)). Therefore $d^* = d$ (by Lemma 4.4.2(4)). So $d^* \in D_j^m$ (since $d \in D_j^m$). But $d^* \notin D_j^m$. Thus $\eta_{\lambda X,i_0}(d^*) \in (\langle R \rangle \lambda_P(e^j))^{d_j^*}$.

So we have proved that $\eta_{\lambda X,i_0}(d^*) \in (\langle R \rangle \lambda_P(e^j))^{d_j^*}$ for every $j < m$. Therefore we conclude that $\eta_{\lambda X,i_0}(d^*) \in \eta_{\lambda P,i_0}(d^*)$. □

Theorem 4.4.15. *The MBA* $\mathbf{P}_{\mathcal{F}}$ *is freely generated by* $\{p_0, \ldots, p_{r-1}\}$.

Proof. By Lemma 4.4.13 and Lemma 4.4.14, every one-element subset of W is expressible via $\{p_0, \ldots, p_{r-1}\}$. Hence every subset of W is expressible via $\{p_0, \ldots, p_{r-1}\}$. Therefore $\{p_0, \ldots, p_{r-1}\}$ generates the MBA $\mathbf{P}_{\mathcal{F}}$.

Now let $(\mathbf{M}, E, \exists)$ be an MBA and $f_0 : \{p_0, \ldots, p_{r-1}\} \to \mathbf{M}$. Then let $(\mathbf{M}_0, E, \exists)$ be the MBA-subalgebra of $\mathbf{M}$ generated by $\{f_0(p_0), \ldots, f_0(p_{r-1})\}$. By Lemma 4.4.13 and Lemma 4.4.14, the assumptions of Theorem 4.3.2 are satisfied vacuously. Hence f_0 can be extended to an MBA-homomorphism $f : \mathbf{P}_{\mathcal{F}} \to \mathbf{M}_0$. So f_0 can be extended to an MBA-homomorphism from $\mathbf{P}_{\mathcal{F}}$ into $\mathbf{M}$.

Thus the MBA $\mathbf{P}_{\mathcal{F}}$ is freely generated by $\{p_0, \ldots, p_{r-1}\}$. □

So the complex algebra of the bounded graph on page 106 is an MBA freely generated by the empty set (see page 112 as well) and the complex algebra of the bounded graph on page 108 is an MBA freely generated by one element $p_0 = \{w_0, w_1, w_4, w_5, w_6, w_7\}$ (see page 112 as well).

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