IAEE 30th Conference
Wellington, New Zealand 18-21 February 2007

Electricity Market Games with Constraints on Transmission Capacity and Emissions

Javier Contreras* Jacek B. Krawczyk† James Zuccollo‡

February 2007

Abstract

Consider an electricity market populated by competitive agents using thermal generating units. Such generation involves the emission of pollutants, on which a regulator might impose constraints. Transmission capacities for sending energy may naturally be restricted by the grid facilities. Both pollution standards and transmission capacities can impose several constraints upon the joint strategy space of the agents. We propose a coupled constraints equilibrium as a solution to the regulator's problem of avoiding both congestion and excessive pollution. Using the coupled constraints Lagrange multipliers as taxation coefficients the regulator can compel the agents to obey the multiple constraints. However, for this modification of the players' payoffs to induce the required behaviour a coupled constraints equilibrium needs to exist and be unique. A three-node bilateral market example with a DC model of the transmission line constraints described in [2] possesses these properties and will be used in this paper to discuss and explain the behaviour agents subjected to coupled constraints.

1 Introduction

The aim of this paper is to examine the impact of pollution standards on electricity generators already subjected to grid facility restrictions.

We consider an electricity market populated by competitive agents using thermal generating units. Such generation produces the emission of pollutants, on which a regulator might impose constraints. Transmission capacities for sending energy may naturally be restricted by the grid facilities. Both pollution standards and transmission capacities can be defined as constraints upon the joint strategy space of the agents. We propose a *coupled constraints equilibrium* as a solution concept for this problem, see [11], [7], [2], [5].

*Universidad de Castilla–La Mancha, Spain
†Victoria University of Wellington, New Zealand
‡Victoria University of Wellington, New Zealand; supported by VUW FCA FRG-05 (24644).

Under this solution concept the regulator can compute (for sufficiently concave games) the generators' outputs that are both unilaterally non-improvable (Nash) and satisfying the constraints imposed on the joint strategy space. If the regulator can impose penalties on the generators for the joint constraints' violation then the game becomes "decoupled" and the players implement the *coupled constraints equilibrium* to avoid fines and produce outputs avoiding both congestion and excessive pollution.

The penalties that prevent the generators from excessive production are computed using the coupled constraints Lagrange multipliers. However, for this modification of the players' payoffs to induce the required behaviour a coupled constraints equilibrium needs to exist and be unique. A three-node bilateral market example with a DC model of the transmission line constraints described in [2] possesses these properties and will be used in this paper to discuss and explain the behaviour of agents subjected to coupled constraints.

For the results, we use NIRA, which is a piece of software dedicated to min-maximize the Nikaido-Isoda function and thus compute a coupled constraints equilibrium, see [8]. We also notice that a coupled constraints equilibrium could be obtained as a solution to quasi-variational inequality (see [4], [10]).

Following is a brief outline of what this paper contains. In section 2 a bilateral electricity market game is presented, including the transmission and pollution constraints modelling. Section 3 briefly canvasses the idea of a coupled constraints equilibrium and the algorithm that will be used to find them. Section 4 presents the parameters of the case study and the results of our analysis. In section 5 an economic interpretation is given to the results. The concluding remarks summarise our findings.

2 A bilateral electricity market game

2.1 A game with constraints

An electricity market is a system for effecting the purchase and sale of electricity, where the interaction between supply and demand sets the market price. It can be either pool-based or bilateral, or a combination of both. A pool-based market can be understood as an auction where competing generators offer their electricity output at increasing prices and consumer loads bid for it at decreasing prices. Transactions are typically cleared and settled by the market operator or by a special-purpose independent entity charged exclusively with that function. If the offers by the generators and the demand bids are matched bilaterally then the market is known as bilateral. We focus on this type of electricity market in this paper.

Transmission systems connect generators and consumer loads in an electricity network and they are operated to allow for continuity of supply. Transmission networks can experience bottlenecks, where generation that may be expensive needs to be dispatched to isolated demands. In addition, the authorities usually establish pollution limits to the generators' emissions. These constraints limit the production of the generators and, consequently, their profits. In the following subsections we explain how the generators optimize their production and how the network and environmental constraints affect their profits in a coupled constraints game.

2.2 Generator's problem

We assume no arbitrage (existence of marketers that can buy and sell power from producers and consumers) and a linear DC representation of the network[1]. Indices $i, j = 1, \ldots, N$ indicate nodes and each company $f = 1, \ldots, F$ owns several generating units $g = 1, \ldots, G$ distributed throughout the network. $C(P_{fgi})$ is the cost per MWh of generating unit g that belongs to company f and is placed at node i; its production is given by P_{fgi} in MW. The maximum capacity of a generator is $\overline{P}_{fgi}$. Consumer loads at node i consume q_i MW of power. At each node, linear demand functions are assumed to be of the form $p_i(q_i) = a_i - (a_i/b_i)q_i$ where price is expressed in \$/MWh and a_i and b_i are the price and quantity intercepts, respectively. It is also assumed that the market is bilateral, and s_{fj} MW are sold by company f to consumers at node j. Market clearing is such that $\sum_f s_{fj} = q_j$. Also, an energy balance is imposed on each company: $\sum_{i,g} P_{fgi} = \sum_j s_{fj}$. Given that, each company f chooses generation P_{fgi} and sales s_{fi} to maximize profit (\$/h), which is equal to revenue minus generation costs:

$$\max \sum_j \left[a_j - \frac{a_j}{b_j} \left(s_{fj} + \sum_{k \neq f} s_{kj} \right) \right] s_{fj} \; - \sum_{i,g} C(P_{fgi}) P_{fgi} \tag{1}$$

subject to:

$$0 \leq P_{fgi} \leq \overline{P}_{fgi}, \qquad \forall \text{ nodes } i, \text{ generators } g \tag{2}$$

$$\sum_j s_{fj} = \sum_{i,g} P_{fgi}, \qquad \text{for each firm } f \tag{3}$$

$$\sum_f s_{fj} = q_j, \qquad \text{for each node } j \tag{4}$$

We are interested in a non cooperative solution to the game at hand. This means that we are looking for a distribution of generation and the corresponding payoffs such that no player can improve his own payoff by a unilateral action. Bearing in mind that the solution is required to satisfy constrains, it will need to be understood as a "generalised" Nash-Cournot equilibrium. We explain this concept in section 3.

2.3 Energy transmission constraints

The generating units and the nodes are connected by transmission lines, forming a network. The lines provide a path to transmit the power produced by the generators to the nodes for consumption. The power flowing through the lines is subject to thermal line limits. These limits are set in both directions of the flow in a line, that is why the absolute value is used in equation (5). To find a unique solution of the power produced by the generators, it is necessary to select a node as the reference

[1]See [3] for technical details

node. This is called the “slack” node or “swing” node (see [3] for details). The following equation expresses the flow going through the lines as a linear function of the power injected in the nodes; this is called the DC approximation of the power flow:

$$\mathbf{P}_{i\to j} \equiv \left[B_d \cdot A^T \cdot B^{-1}\right] \cdot \mathbf{P}, \quad \text{with} \quad |\mathbf{P}_{\mathrm{i\to j}}| \leq \overline{\mathbf{P}}_{\mathrm{i\to j}}, \tag{5}$$

where the variables and parameters are as follows:

$\mathbf{P}_{i\to j}$ is a column vector whose dimension is equal to the number of lines of the network. Each element represents the flow through the line $i - j$, measured with respect to a predefined base power of the system. The flow measured w.r.t. to the base power is a per-unit (p.u.) flow[2].

B_d is the diagonal branch susceptance matrix, whose number of rows and columns is equal to the number of lines in the network. The diagonal terms of B_d are the inverses of reactances (a.k.a. susceptances) of the lines, where the reactances are in p.u. with respect to a predefined base impedance of the system.

A is the node-branch incidence matrix; its dimensions are the number of nodes minus one (slack node) times the number of lines. The values of A are equal to $+1$ if the line $i - j$ starts at node i, and -1 if it ends at node j.

B is the diagonal node-to-node susceptance matrix; its dimensions are equal to the number of nodes minus one (slack node). The diagonal terms b_{ii} are equal to the sum of the susceptances of the lines that are connected to node i, and the terms b_{ij} are equal to the negative of the susceptances of the lines that connect node i and node j.

$\mathbf{P}$ is a column vector whose dimension is equal to the number of nodes minus one (the slack node). Its terms are of the form $(P_{fgi} - q_i)$, representing the p.u. power injected (generation minus demand) in each of the nodes, except the slack node.

$\overline{\mathbf{P}}_{i\to j}$ is a column vector whose dimension is equal to the number of lines of the network. Each of its elements represents the thermal limit (maximum active power that can flow through a line) of a line in p.u.

2.4 Pollution constraints

The generation of electricity releases several contaminants into the atmosphere. The overall goal of reducing the emission of pollutants has to be expressed as a constraint for the overall production of all generating units. There are three main types of emissions: CO_2, SO_2 and NO_x. The general expression of the emission constraints per type of emission is such that:

[2]See [12] for details of the p.u. calculation.

$$\sum_{i=1}^{n}(\alpha_{i\ell} + \beta_{i\ell} \cdot P_{fgi} + \gamma_{i\ell} \cdot P_{fgi}^2) \leq E_\ell, \tag{6}$$

where P_{fgi} is the output of generating unit g that belongs to company f and is placed at node i. Coefficients $\alpha_{i\ell}$, $\beta_{i\ell}$, and $\gamma_{i\ell}$ correspond to generator i for emission type ℓ and E_ℓ is the maximum allowed emission of type ℓ, usually measured in lb/h or Ton/h.

3 Generalised Nash equilibria

A generalised Nash equilibrium problem (GNEP) is an extension of a standard Nash equilibrium problem in which players' strategy sets are allowed to depend upon other players' strategies. The competition between electricity generating firms subject to constraints described above in section 2 is an example of such a problem. Analytical solutions to GNEPs are not normally possible so section 3.2 describes a numerical method for solving some such problems.

3.1 Coupled constraints equilibria

The problem described in section 2 involves an action space that is jointly restricted for all players in the game. This is termed a coupled constraints game and the solution of such a game is known as a coupled constraints equilibrium (CCE). A CCE is a refinement of a Nash equilibrium and is particularly useful in a class of problems where competing agents are subjected to regulation. Many electricity market and environmental problems belong to this class. CCE allows modelling of a situation in which the actions of one player condition how 'big' the actions of other players can be. Constraints in which the actions of one player do not affect the action space of another (as in Nash equilibrium problems) are called uncoupled.

In our problem there are two such sets of coupled constraints: the line constraints and the environmental constraints. In both cases a limit is placed on a measurable variable — the flow of electricity through a particular line or the ambient pollution levels — and the actions of the players are constrained to jointly satisfy these limits.

In these games the constraints are assumed to be such that the resulting collective action set X is a closed convex subset of $\mathbb{R}_m$. If X_i is player-i's action set, $X \subseteq X_1 \times \cdots \times X_n$ is the collective action set where $X = X_1 \times \cdots \times X_n$ represents the special case in which the constraints are uncoupled.

Consider the solution to this game represented by the collective action $\mathbf{x}^*$ where players' payoff functions, ϕ_i, are continuous in all players' actions and concave in their own action. The Nash equilibrium can be written as

$$\phi_i(\mathbf{x}^*) = \max_{\mathbf{x}\in X} \phi_i(y_i|\mathbf{x}^*) \tag{7}$$

where $y_i|\mathbf{x}$ denotes a collection of actions where the ith agent "tries" y_i while the remaining agents continue to play x_j, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$. At $\mathbf{x}^*$ no player can improve his own payoff through a unilateral change in his strategy so $\mathbf{x}^*$ is a

Nash equilibrium point. If X is a closed and strictly convex set defined through coupled constraints (like (6)) then $\mathbf{x}^*$ is a CCE.

3.2 NIRA

As said above, games with coupled constraints rarely allow for an analytical solution and so numerical methods must be employed. Here we use a method based on the Nikaido-Isoda function and a relaxation algorithm (hence the name: NIRA).

3.2.1 The Nikaido-Isoda function

This function is a cornerstone or the NIRA technique for solving games for their CCE. It transforms the complex process of solving a (constrained) game into a far simpler (constrained) optimisation problem.

Definition 3.1. *Let ϕ_i be the payoff function for player i and X a collective strategy set as before. The* Nikaido-Isoda *function $\Psi : X \times X \to I\!R$ is defined as*

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} [\phi_i(y_i|\mathbf{x}) - \phi_i(\mathbf{x})] \tag{8}$$

Result 3.1. *See [13].*

$$\Psi(\mathbf{x}, \mathbf{x}) \equiv 0 \qquad\qquad \mathbf{x} \in X. \tag{9}$$

Each summand from the Nikaido-Isoda function can be thought of as the improvement in payoff a player will receive by changing his action from x_i to y_i while all other players continue to play according to $\mathbf{x}$. Therefore, the function represents the sum of these improvements in payoff. Note that the *maximum* value this function can take, for a given $\mathbf{x}$, is always nonnegative, owing to Result 3.1 above. The function is everywhere non-positive when either $\mathbf{x}$ or $\mathbf{y}$ is a Nash equilibrium point, since in an equilibrium situation no player can make any more improvements to their payoff. Consequently, each summand in this case can be at most zero at the Nash equilibrium point [7].

When the Nikaido-Isoda function cannot be made (significantly) positive for a given $\mathbf{y}$, we have (approximately) reached the Nash equilibrium point. This observation is used to construct a termination condition for the relaxation algorithm. An ε is chosen such that, when

$$\max_{\mathbf{y} \in \mathbb{R}^m} \Psi(\mathbf{x}^s, \mathbf{y}) < \varepsilon, \tag{10}$$

(where $\mathbf{x}^s$ is the s-th iteration approximation of x^*) the Nash equilibrium would be achieved to a sufficient degree of precision [7].

The Nikaido-Isoda function is used to construct the optimum response function. This function is similar to the best response function in standard non-cooperative game theory. It defines each player's optimal action to maximise his payoff given what the other players have chosen. The vector $Z(\mathbf{x})$ gives the 'best move' of each player when faced with the collective action $\mathbf{x}$. It is at this point that the coupled constraints are introduced into the optimisation problem. The maximisation of the Nikaido-Isoda function in equation (11) is performed subject to the constraints on the players' actions.

Definition 3.2. *The* optimum response function *at point* $\mathbf{x}$ *is*

$$Z(\mathbf{x}) \in \arg\max_{\mathbf{y}\in X} \Psi(\mathbf{x}, \mathbf{y}). \tag{11}$$

3.2.2 The relaxation algorithm

The relaxation algorithm iterates to find the Nash equilibrium of a game. It starts with an initial estimate of the Nash equilibrium and iterates from that point towards $Z(\mathbf{x})$ until no more improvement is possible. At such a point every player is playing their optimum response to every other player's action and the Nash equilibrium is reached. The relaxation algorithm, when $Z(\mathbf{x})$ is single-valued, is

$$\mathbf{x}^{s+1} = (1-\alpha_s)\mathbf{x}^s + \alpha_s Z(\mathbf{x}^s) \qquad \begin{array}{l} 0 < \alpha_s \leq 1 \\ s = 0, 1, 2, \ldots \end{array} \tag{12}$$

¿From the initial estimate, an iterate step $s+1$ is constructed by a weighted average of the players' improvement point $Z(\mathbf{x}^s)$ and the current action point $\mathbf{x}^s$. Given concavity assumptions explained in section 3.3, this averaging ensures convergence (see [13], [7]) to the Nash equilibrium by the algorithm. By taking a sufficient number of iterations of the algorithm, the Nash equilibrium $\mathbf{x}^*$ can be determined with a specified precision.

3.3 Existence and uniqueness of equilibrium points

It is one thing to know that one has a method to solve games with constraints but, before proceeding, one needs to establish that the game has an equilibrium at all. Furthermore, since the NIRA algorithm converges to a single equilibrium point it would be nice if that equilibrium could be shown to be unique. The conditions for existence and uniqueness for games with coupled constraints are more intricate than those for games with uncoupled constraints. It is known that every concave n-person game with uncoupled constraints has an equilibrium point [11]. The equivalent definition for a game with coupled constraints relies upon the notion of a weakly convex-concave function.

A weakly convex-concave function is a bivariate function that exhibits weak convexity in its first argument and weak concavity in its second argument. The next three definitions (see [9] or [13]) formalise this notion.[3] As Theorem 3.1 (the convergence theorem) will document, weak convex-concavity of a function is a crucial assumption needed for convergence of a relaxation algorithm to a coupled constraints equilibrium.

[3]Recall the following elementary definition: a function is "just" *convex* $\iff$

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \ \geq \ f\Big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\Big), \quad \alpha \in [0,1].$$

Let X be a convex closed subset of the Euclidean space $\mathbb{R}^m$ and f a continuous function $f : X \to \mathbb{R}$.

Definition 3.3. *A function of one argument $f(\mathbf{x})$ is* weakly convex *on X if there exists a function $r(\mathbf{x}, \mathbf{y})$ such that $\forall \mathbf{x}, \mathbf{y} \in X$*

$$\begin{aligned} &\alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \geq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha)r(\mathbf{x}, \mathbf{y}) \\ &0 \leq \alpha \leq 1, \ \text{ and } \frac{r(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \ \text{ as } \|\mathbf{x}-\mathbf{y}\| \to 0 \qquad \forall \mathbf{x} \in X. \end{aligned} \tag{13}$$

Definition 3.4. *A function of one argument $f(\mathbf{x})$ is* weakly concave *on X if there exists a function $\mu(\mathbf{x}, \mathbf{y})$ such that, $\forall \mathbf{x}, \mathbf{y} \in X$*

$$\begin{aligned} &\alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \leq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha)\mu(\mathbf{x}, \mathbf{y}) \\ &0 \leq \alpha \leq 1, \ \text{ and } \frac{\mu(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \quad \text{ as } \quad \|\mathbf{x}-\mathbf{y}\| \to 0 \qquad \forall \mathbf{x} \in X. \end{aligned} \tag{14}$$

Example: *The convex function $f(x) = x^2$ is weakly concave (see [7])*

Now take a bivariate function $\Psi : X \times X \to \mathbb{R}$ defined on a product $X \times X$, where X is a convex closed subset of the Euclidean space $\mathbb{R}^m$.

Definition 3.5. *A function of two vector arguments, $\Psi(\mathbf{x}, \mathbf{y})$ is referred to as* weakly convex-concave *if it satisfies weak convexity with respect to its first argument and weak concavity with respect to its second argument.*

The functions $r(\mathbf{x}, \mathbf{y}; \mathbf{z})$ and $\mu(\mathbf{x}, \mathbf{y}; \mathbf{z})$ were introduced with the concept of weak convex-concavity and are called the *residual terms*. Notice that smoothness of $\Psi(\mathbf{z}, \mathbf{y})$ is not required. However, if $\Psi(\mathbf{x}, \mathbf{y})$ is twice continuously differentiable with respect to both arguments on $X \times X$, the residual terms satisfy (see [7])

$$r(\mathbf{x}, \mathbf{y}; \mathbf{y}) = \tfrac{1}{2}\langle A(\mathbf{x}, \mathbf{x})(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + o_1(\|\mathbf{x}-\mathbf{y}\|^2) \tag{15}$$

and

$$\mu(\mathbf{y}, \mathbf{x}; \mathbf{x}) = \tfrac{1}{2}\langle B(\mathbf{x}, \mathbf{x})(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + o_2(\|\mathbf{x}-\mathbf{y}\|^2) \tag{16}$$

where $A(\mathbf{x}, \mathbf{x}) = \Psi_{\mathbf{xx}}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido-Isoda function with respect to the first argument and $B(\mathbf{x}, \mathbf{x}) = \Psi_{\mathbf{yy}}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido-Isoda function with respect to the second argument, both evaluated at $\mathbf{y} = \mathbf{x}$.

To prove the inequality of condition (5) of Theorem 3.1 (the convergence theorem, below) under the assumption that $\Psi(\mathbf{x}, \mathbf{y})$ is twice continuously differentiable, it suffices to show that

$$Q(\mathbf{x}, \mathbf{x}) = A(\mathbf{x}, \mathbf{x}) - B(\mathbf{x}, \mathbf{x}) \tag{17}$$

is strictly positive definite.

Theorem 3.1 (Convergence theorem)**.** *There exists a unique normalised Nash equilibrium point to which the algorithm* (12) *converges if:*

1. *X is a convex, compact subset of $\mathbb{R}^m$,*

2. *the Nikaido-Isoda function* $\Psi : X \times X \to I\!R$ *is a weakly convex-concave function and* $\Psi(\mathbf{x},\mathbf{x}) = 0$ *for* $\mathbf{x} \in X$ *,*

3. *the optimum response function* $Z(\mathbf{x})$ *is single valued and continuous on* X*,*

4. *the residual term* $r(\mathbf{x},\mathbf{y};\mathbf{z})$ *is uniformly continuous on* X *w.r.t.* $\mathbf{z}$ *for all* $\mathbf{x},\mathbf{y} \in X$ *,*

5. *the residual terms satisfy*

$$r(\mathbf{x},\mathbf{y};\mathbf{y}) - \mu(\mathbf{y},\mathbf{x};\mathbf{x}) \geq \beta(\|\mathbf{x}-\mathbf{y}\|)\,, \qquad \mathbf{x},\mathbf{y} \in X \tag{18}$$

   where $\beta(0) = 0$ *and* β *is a strictly increasing function (*i.e., $\beta(t_2) > \beta(t_1)$ *if* $t_2 > t_1$*),*

6. *the relaxation parameters* α_s *satisfy*

   - *either (non-optimised step)*

     (a) $\alpha_s > 0$*,*
     (b) $\sum_{s=0}^{\infty} \alpha_s = \infty$*,*
     (c) $\alpha_s \to 0$ *as* $s \to \infty$*.*

   - *or (optimised step)*

$$\alpha_s = \arg \min_{\alpha \in [0,1)} \left\{ \max_{\mathbf{y} \in X} \Psi(\mathbf{x}^{(s+1)}(\alpha),\mathbf{y}) \right\}. \tag{19}$$

Proof. See [7] for a proof. □

3.4 Enforcement through taxation

Once a CCE, x^*, has been computed it is possible to create an unconstrained game which has x^* as its solution by a simple modification to the players' payoff functions. For example, a regulator may compute that x^* is the CCE of a game involving the desired constraints on agents' behaviour. He may then wish to induce the players to arrive at this point through a scheme of taxation to modify their payoff functions. This can be achieved by the use of penalty functions that punish players for breaching the coupled constraints.

Penalty functions are weighted by the Lagrange multipliers obtained from the constrained game. For each constraint, player f is taxed according to the function

$$T_{f\ell}(\lambda,\mathbf{x}) = \lambda_\ell \max(0, E_\ell(\mathbf{x}) - K_\ell) \tag{20}$$

where λ_ℓ is the Lagrange multiplier associated with the ℓth constraint, $E_\ell(\mathbf{x}) \leq K_\ell$, and $\mathbf{x}$ is the vector of players' actions.

The players' payoff functions, so modified, will be

$$\overline{\phi}_f(\mathbf{x}) = R_f(\mathbf{x}) - C_f(\mathbf{x}) - \sum_\ell T_{f\ell}(\lambda,\mathbf{x}) \tag{21}$$

where R_f and C_f are firm f's revenue and cost functions respectively. Notice that under this taxation scheme the players pay no penalty fee if all constraints are satisfied.

The Nash equilibrium of the new unconstrained game with payoff functions $\overline{\phi}$ is implicitly defined by the equation

$$\overline{\phi}(\mathbf{x}^{**}) = \max_{y_f \in \mathrm{I\!R}^+} \overline{\phi}(y_f|\mathbf{x}^{**}) \qquad \forall\, f, \tag{22}$$

(compare with equation (7)). For the setup of the problem considered in this paper $x^* = x^{**}$. That is, the CCE is equal to the unconstrained equilibrium with penalty functions for breaches of the 'constraints' weighted by the Lagrange multipliers (see [7], [5] and [6] for a more detailed discussion).

4 Results

A suite of MATLAB routines called NIRA has been designed to compute coupled constraints equilibria ([8], [1]). The results reported here were obtained using NIRA.

4.1 The model

4.1.1 Without coupled constraints

The example is taken from [2]. Numerical data for the general formulation of problem is as follows. There are three nodes, $i = 1,\ 2,\ 3$, each of which has customers. Generation only occurs at nodes 1 and 2 and each pair of nodes is connected by a single transmission line. The demand functions are $p_i(q_i) = 40 - 0.08q_i$, for nodes $i = 1,\ 2$, and $p_3(q_3) = 32 - 0.0516q_3$ \$/MWh. Thus, the demand is more elastic at the demand-only node 3. Firm's 1 generator is placed at $i = 1$ and firm's 2 at $i = 2$. Since each firm has only one generator we drop the g subscript for brevity (*e.g.*, P_{fgi} becomes P_{fi}). Both generators have unlimited capacity and constant marginal costs such that $C(P_{1,i}) = 15$ for firm 1 and $C(P_{2,i}) = 20$ for firm 2. Marginal costs are measured in \$/MWh. The three lines have equal impedances of 0.2 p.u. and the base power is 100 MVA. The slack node is node 3.

As a result, both firms solve the following optimization problems, based upon equation (1), subject to constraints and coupled constraints (2)–(5):

Firm 1:

$$\max\Big\{[40 - 0.08q_1]s_{11} + [40 - 0.08q_2]s_{12} + [32 - 0.0516q_3]s_{13} - 15P_{1,1}\Big\} \tag{23}$$

Firm 2:

$$\max\Big\{[40 - 0.08q_1]s_{21} + [40 - 0.08q_2]s_{22} + [32 - 0.0516q_3]s_{23} - 20P_{2,2}\Big\} \tag{24}$$

subject to:

$$\begin{aligned} P_{1,1} &= s_{11} + s_{12} + s_{13}, && \text{(25a)} \\ P_{2,2} &= s_{21} + s_{22} + s_{23}, && \text{(25b)} \\ q_1 &= s_{11} + s_{21}, && \text{(25c)} \\ q_2 &= s_{12} + s_{22}, && \text{(25d)} \\ q_3 &= s_{13} + s_{23}, && \text{(25e)} \\ \text{all } & s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23}, \overline{P}_{1\to 2}, \overline{P}_{1\to 3}, \overline{P}_{2\to 3} \text{ nonnegative}, && \text{(25f)} \end{aligned}$$

where the decision variables of the generators (firms) are: s_{11}, s_{12} and s_{13} for the first generator and s_{21}, s_{22} and s_{23} for the second generator. The remaining variables are dependent on the decision variables. Part of the solution will constitute the Lagrange multipliers that a regional regulator will be able to use to enforce the equilibrium (see section 3.4).

4.1.2 Transmission line constraints

A constraint on transmission line capacity is imposed as described in equation (5)[4]. The equation of the constraint in this example is

$$\left| \left[B_d \cdot A^T \cdot B^{-1} \right] \cdot \begin{pmatrix} P_{1,1} - q_1 \\ P_{2,2} - q_2 \end{pmatrix} \right| \leq \begin{pmatrix} \overline{P}_{1\to 2} \\ \overline{P}_{1\to 3} \\ \overline{P}_{2\to 3} \end{pmatrix}, \tag{26}$$

The values of the transmission line constraints are as follows:

$$B_d = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 10 & -5 \\ -5 & 10 \end{pmatrix}. \tag{27}$$

Note that the first row of matrix A (whose dimension is: 2 nodes times 3 lines) expresses that node 1 is the starting node of lines 1–2 and 1–3, and the second row means that node 2 is the ending node of line 1–2 and the starting node of line 2–3. Node 3 is the slack node for which there are no calculations, since it is the reference node.

The diagonal terms of B are computed as follows: b_{11} is the sum of the two line susceptances connected to node 1, likewise for the other diagonal term corresponding to node 2. The off-diagonal terms are the susceptances of the lines 1–2 and 2–1 (which is the same line), respectively.

We get this numerical expression for the line constraints:

$$\left[B_d \cdot A^T \cdot B^{-1} \right] = \begin{pmatrix} 0.33 & -0.33 \\ 0.66 & 0.33 \\ 0.33 & 0.66 \end{pmatrix}. \tag{28}$$

[4] For example, the first row of $P_{i\to j}$ is $P_{1\to 2} = 0.66\, s_{12} + 0.33\, s_{13} - 0.66\, s_{21} - 0.33\, s_{23}$. This indicates that the flow along the line from node 1 to node 2 depends not only upon the quantity that is sold to nodes 1 and 2 but also upon the quantity of electricity that is sold to node 3.

4.1.3 Environmental constraints

In this case study, a CO_2 emission constraint is added to the problem formulation. As a result, the problem is set as in (23)–(25), where both firms solve their optimization problems as in (23) and (24), respectively, but, in this case, a new environmental constraint is added to the constraint set (25), so that:

$$(20 - 0.4 \cdot P_{1,1} + 0.004 \cdot P_{1,1}^2) + (22 - 0.3 \cdot P_{2,2} + 0.005 \cdot P_{2,2}^2) \leq 250, \tag{29}$$

where the maximum allowed CO_2 emission is 250 lb/h.

4.2 Base case

Here the firms' problem described by equations (23)–(25) is solved using NIRA. The results of the relaxation algorithm are shown in Table I. The quantities demanded are: $q_1 = 187.5$ MW, $q_2 = 187.5$ MW, and $q_3 = 187.3$ MW. Prices at the nodes, according to the linear demand functions, are: 25, 25, and 22.3 $/MWh. The flows through the lines are: 73.95 MW, 130.65 MW, and 56.7 MW for lines 1–2, 1–3 and 2–3, respectively. Profits for firms 1 and 2 are 3542.1 and 730.6 $/h, respectively. The following Lagrange multipliers are computed: $\lambda_{L1} = 0,\ \lambda_{L2} = 0,\ \lambda_{L3} = 0$.

The three Lagrange multipliers correspond to the three constraints of the flow limits matrix inequality. Since no line constraint is binding, the value of the Lagrange multipliers is equal to zero, as expected.

Table I: Generation and sales in the base case

Sales by Firm 1 (MWh)			Sales by Firm 2 (MWh)			Generation by Firms (MWh)	
s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	$P_{1,1}$	$P_{2,2}$
125	125	142.1	62.5	62.5	45.2	392.1	170.2

4.3 Generation under transmission constraints

For this example, a limit of 25 MW in the transmission capacity of the line that connects nodes 1 and 2 is imposed. As a result, the problem is set as in section 4.2 above with the addition of the constraint described in equation (26). In this case, the thermal limit of line 1–2 is set to 25 MW (i.e. $\overline{P}_{1-2} = 0.25$ p.u.) since the base power is 100 MVA.

This is a game with coupled constraints with two non identical players. We notice that player 1, located at node 1, produces electricity more cheaply than player 2 at node 2. This game satisfies the hypotheses of the convergence theorem, see [2]. The game's solution is obtained using the NIRA software. This will be a combination of the decision variables' values such that the constraints will be satisfied and no player will be able to improve his payoff by an unilateral move.

The results of the relaxation algorithm with a constant step size of 0.5 are shown in Tables II. The quantities demanded are: $q_1 = 199.1$ MW, $q_2 = 175.9$ MW, and

$q_3 = 187.3$ MW. Prices at the nodes, according to the linear demand functions, are: 24.1, 25.9, and 22.3 \$/MWh. The flows through the lines are: 25 MW (line flow limit), 106.15 MW, and 81.15 MW for lines 1–2, 1–3 and 2–3, respectively. Profits for firms 1 and 2 are 2985 and 956.9 \$/h, respectively. The following Lagrange multipliers are computed: $\lambda_{L1} = 4.18,\ \lambda_{L2} = 0,\ \lambda_{L3} = 0$.

The three Lagrange multipliers correspond to the three constraints of the flow limits matrix inequality. Only the first value is non-zero, since the 25 MW line limit is binding.

Table II: Generation and sales with a line flow limit of 25 MW in line 1–2

Sales by Firm 1 (MWh)			Sales by Firm 2 (MWh)			Generation by Firms (MWh)	
s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	$P_{1,1}$	$P_{2,2}$
113.4	101.8	115.1	85.7	74.1	72.2	330.3	232

4.4 Generation under environmental constraints

Here the environmental constraint of equation (29) is included in the problem but the transmission constraint of equation (26) is not. The results obtained are shown in Table III. The quantities demanded are: $q_1 = 139.05$ MW, $q_2 = 139.05$ MW, and $q_3 = 112.22$ MW. Prices at nodes, according to the linear demand functions, are: 28.9, 28.9, and 26.2 \$/MWh, respectively. The flows through the lines are: 38.66 MW, 75.44 MW, and 36.78 MW for lines 1–2, 1–3 and 2–3, respectively. Profits for firms 1 and 2 are 3295.6 and 1135.4 \$/h, respectively. The following Lagrange multipliers are computed: $\lambda_{L1} = 0,\ \lambda_{L2} = 0,\ \lambda_{L3} = 0,\ \lambda_{E1} = 4.31$.

Table III: Generation and sales with a CO_2 emission constraint

Sales by Firm 1 (MWh)			Sales by Firm 2 (MWh)			Generation by Firms (MWh)	
s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	$P_{1,1}$	$P_{2,2}$
85.9	85.9	81.4	53.2	53.2	30.8	253.2	137.2

4.5 Generation under transmission and environmental constraints

In this last case study, both the 25 MW thermal limit of line 1–2 (equation (26)) and the CO_2 emission constraints (equation (29)) are added to the problem formulation.

The results obtained are shown in Table IV. The quantities demanded are: $q_1 = 147.4$ MW, $q_2 = 133.1$ MW, and $q_3 = 114.1$ MW. Prices at nodes, according to the linear demand functions, are: 28.2, 29.4, and 26.1 \$/MWh, respectively. The flows through the lines are: 25 MW (line flow limit), 69.53 MW, and 44.53

MW for lines 1–2, 1–3 and 2–3, respectively. Profits for firms 1 and 2 are 3125.8 and 1235.9 \$/h, respectively. The following Lagrange multipliers are computed: $\lambda_{L1} = 2.58,\ \lambda_{L2} = 0,\ \lambda_{L3} = 0,\ \lambda_{E1} = 4.11$.

Table IV: Generation and sales with a line flow limit of 25 MW in line 1–2 and a CO_2 emission constraint

Sales by Firm 1 (MWh)			Sales by Firm 2 (MWh)			Generation by Firms (MWh)	
s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	$P_{1,1}$	$P_{2,2}$
86.3	79.1	76.5	61.1	54.0	37.5	241.9	152.6

5 Economics of constraints

5.1 Line constraint

The line constraint enforced in this model only restricts flow along the line between the nodes at which the two generators reside. The substantive difference between these two generators is their marginal cost: firm 1's generator has a marginal cost of \$15/MWh while firm 2's generator has a marginal cost of \$20/MWh. Given their respective payoff functions the two firms play a Cournot game at each node.

Since firm 1 has a lower marginal cost it is to be expected that it supplies more of the market at each node than firm 2, and this is borne out by the numerical results presented above. However, the line flow constraint effectively restricts the quantity of electricity that firm 1 can supply to node 2. This shelters firm 2 from competition[5] at node 2 and allows it to "dominate" the market at that node *i.e.*, increase its sales there see Table II) and, consequently, increase its profits.

As firm 2 increases its sales at node 2 it also increases its total generation. Concurrently, firm 1's sales at node 2 decrease and with the decrease in sales comes a decrease in generation. Since firm 2 is now providing a greater proportion of the generation capacity on the grid, firm 2's share of the market at nodes 1 and 3 also increases. This effect can be clearly seen in figure 1 where firm 2's sales increase at all nodes as the transmission constraint tightens. The converse is true for firm 1's sale at all nodes.

Observe from the data that, with no constraints, the generation ratio between firm 2 and firm 1 is $170.2/392.1 = 0.43$. Introducing the transmission constraint increases this ratio to 0.70 which favours firm 2 as described above. Similarly, firm 2's share of total profits increases from 0.20 to 0.32 with the introduction of the line constraint. It is apparent that the transmission constraint distorts the market outcome in favour of the less efficient firm by sheltering it from competition.

[5]We could perform a sensitivity analysis to establish what marginal cost firm 1 would have to have to not allow firm 2 to dominate the market at node 2.

5.2 Emissions constraint

In general, firm 2 is less efficient with a higher level of CO_2 emitted for each unit of electricity generated. However, as this firm produces less electricity than firm 1 (for "economic" reasons) it ends up to be a lesser polluter than its competitor. This means that firm 2 might have a comparative advantage as against firm 1 when the emissions constraint is implemented, as long as its output does not increase "too much". Hence, as the emissions constraint tightens, firm 2 might gain market share and increases its share of total profits.

The results presented support this conjecture. Enforcement of the CO_2 constraint reduces firm 1's total generation by 35% while only reducing firm 2's generation by 19%, see Table IV. Firm 2's share of the profits also increases to 0.34 (from 0.20) when the emissions constraint is introduced. This same thing can be seen in figure 1: the gradient of the plot in the x-z plane is greater for firm 1 than for firm 2. This indicates that, as the CO_2 constraint is tightened (CO_2 limit decreases), firm 1 is forced to reduce sales at a greater rate than firm 2.

5.3 Both constraints simultaneously

When both effects described above apply the net result is that firm 2 gains market share over the case in which there is only an environmental constraint (63% of total generation as compared to 60%). Note that firm 2's market share is still less than when there is only a transmission constraint (70%). This is because the gains are dampened by the lower total generation levels permitted by the environmental constraint. The sales plots of figure 1 show this dampening effect clearly. The tighter the CO_2 limit is, the less effect the transmission constraint has upon the firms' sales (gradient in the y-z plane decreases in the direction of decreasing 'CO_2 limit').

Figure 1 also indicates that the brunt of the sales reduction induced by the tightening of the CO_2 constraint is borne by the firm which is favoured by the *current* state of the transmission constraint. Firm 1's sales decrease faster as the environmental constraint tightens when the transmission constraint is relaxed (and firm 1 can compete freely at node 2). Conversely, firm 2's sales decrease faster as the environmental constraint tightens *when* the line flow limit is very small (and firm 2 is sheltered from competition at node 2).

These results show that, even in the presence of a transmission constraint, the introduction of an environmental constraint may benefit the less environmentally friendly firm. Reducing the total generation through environmental constraints also appears to mitigate some of the market distortion introduced by the transmission constraint.

5.4 Welfare implications

The results presented here do not include calculations of consumer surplus and so cannot fully describe welfare. However, some conclusions can still be drawn from the manner in which the sum of payoffs changes as the constraints' limits are varied. Figure 2 shows plots of each firm's profits and total profits as the constraints vary.

The shape of the individual firm's plots is explained by the previous discussion of the effect of the individual constraints. Here an explanation is offered for the shape of the plot of total profits.

As the line limit is tightened total profit drops. This is because firm 2 contributes a greater share of total generation as the line 1-2 limit is decreased and firm 2 has a higher marginal cost of generation than firm 1. However, tightening the CO_2 constraint has more interesting results.

Unlike the introduction of a transmission constraint, the introduction of an environmental constraint reduces the total generation level. Because total generation is reduced, the price rises at all nodes and total profit initially increases. As the CO_2 limit is tightened the total level of generation moves from a Cournot duopoly level towards the monopoly level and profits rise accordingly. Firm 2's share of generation expands and the overall average cost of generation rises also (since firm 2's marginal cost is greater than firm 1's). There comes a point at which the rising average cost of production and the decreasing sales of the firms overcome the effect of rising prices and total profits begin to drop. It is at this point that the CO_2 constraint is forcing the firms to jointly produce at below the monopoly level.

While total profits might increase when an environmental constraint is first applied the same cannot be assumed of welfare. Numbers are not presently available but it is likely that the increasing prices will cause consumer surplus to decrease. Some of the profits are likely to be returned to consumers in the form of higher wages but our partial equilibrium analysis excludes such considerations.

Presumably the environmental constraint will be set "optimally", such that it maximises some welfare function that includes the cost of pollution. Our study shows that it is possible that this optimal constraint will increase the profit of the polluting firms and shift surplus away from consumers. The total change in welfare is unknown but the possibility of this result seems inequitable at least.

6 Concluding remarks

We have proposed a methodology for the analysis of the impact of various constraints on electricity generation. In particular, this analysis should be useful for a regional government that is interested in an assessment of electricity supply changes due to an introduction of environmental constraints. For the case study considered in the paper, we notice the possibility of some market distortion when transmission constraints exist.

Introduction of an environmental constraint, which many businesses fear, may actually increase the business profits and make the consumers worse off economically.

We believe that thanks to our analysis, the regional government's choices will be informed by the tradeoffs between constraint satisfaction, economic activity and electricity supply.

In the next paper we will fully address the issue of regional welfare changes due to constraints by analysing the consumers' surplus. We might also explore the possibility of extending the analysis to include the optimal choice of environmental constraints.

References

[1] S. Berridge and J. B. Krawczyk, *Relaxation algorithms in finding Nash equilibria*, Economic Working Papers Archive, 1997, URL: http://econwpa.wustl.edu/eprints/comp/papers/9707/9707002.abs.

[2] J. Contreras, M. Klusch, and J. B. Krawczyk, *Numerical solutions to Nash-Cournot equilibria in coupled constraint electricity markets*, IEEE Transactions on Power Systems **19** (2004), no. 1, 195–206.

[3] A. G. Expósito and A. Abur, *Análisis y operación de sistemas de energía eléctrica*, McGraw-Hill-Interamericana de España, 2002.

[4] B. E. Hobbs and J.-S. Pang, *Nash-Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints*, Operations Research **55** (2007), no. 1, 113–127.

[5] J. B. Krawczyk, *Coupled constraint Nash equilibria in environmental games*, Resource and Energy Economics **27** (2005), 157–181.

[6] ______, *Numerical solutions to coupled-constraint (or generalised) Nash equilibrium problems*, Computational Management Science (Online Date: November 09, 2006), http://dx.doi.org/10.1007/s10287–006–0033–9.

[7] J. B. Krawczyk and S. Uryasev, *Relaxation algorithms to find Nash equilibria with economic applications*, Environmental Modelling and Assessment **5** (2000), 63–73.

[8] J. B. Krawczyk and J. Zuccollo, `NIRA-3`: *A MATLAB package for finding Nash equilibria in infinite games*, Working Paper, School of Economics and Finance, VUW, 2006.

[9] E. A. Nurminski, *Progress in nondifferentiable optimization*, ch. Subgradient Method for Minimizing Weakly Convex Functions and ϵ-Subgradient Methods of Convex Optimisation, pp. 97–123, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1982.

[10] J.-S. Pang and M. Fukushima, *Quasi-variational inequalities, generalized Nash equilibria and multi-leader-follower games*, Computational Management Science **1** (2005), 21–56.

[11] J. B. Rosen, *Existence and uniqueness of equilibrium points for concave n-person games*, Econometrica **33** (1965), no. 3, 520–534.

[12] W. D. Stevenson, *Elements of power system analysis*, McGraw-Hill New York, 1982.

[13] S. Uryasev and R. Y. Rubinstein, *On relaxation algorithms in computation of noncooperative equilibria*, IEEE Transactions on Automatic Control **39** (1994), no. 6, 1263–1267.

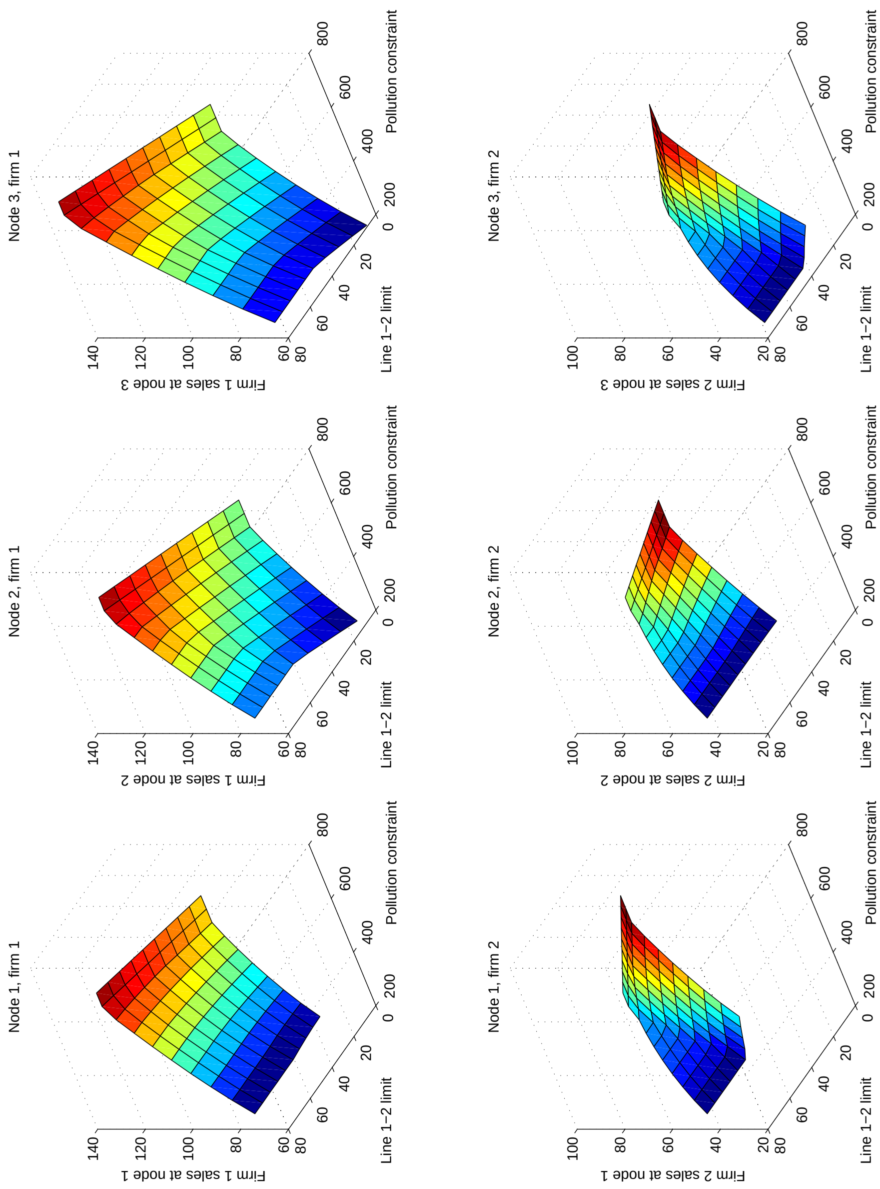

Figure 1: Sales plots.

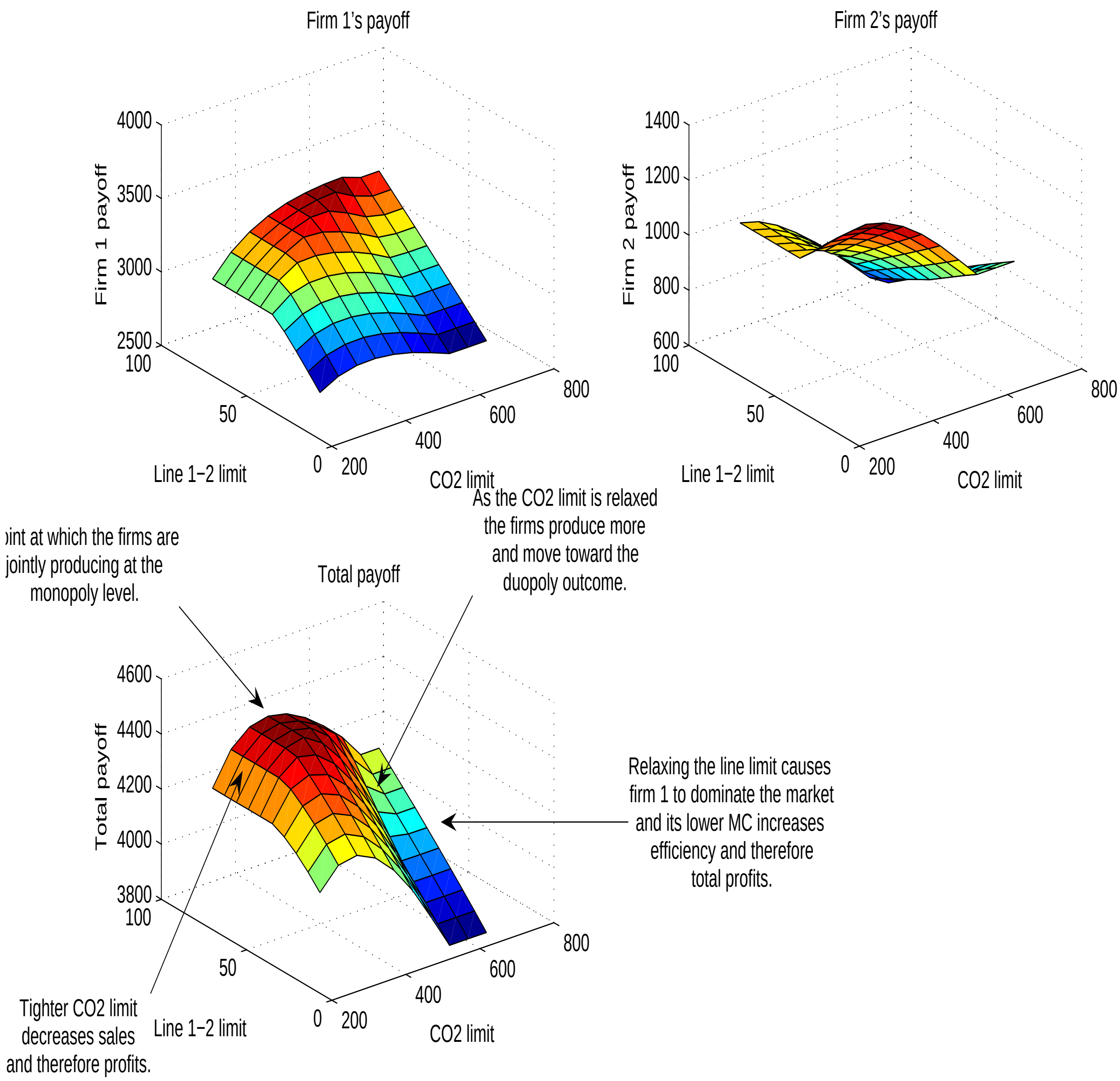

Figure 2: Payoff plots.