

Convertible Bond Pricing with Stochastic Volatility

by

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Contents

1	Introduction.	1
2	Basic options pricing and practical application.	3
3	Foundations. The basic case.	8
3.1	Solving the basic PDE:	14
4	Adding more dimensions into the model.	15
4.1	Multiple state variables.	17
4.2	Application to options pricing with stochastic volatility.	18
4.3	Application to options pricing with stochastic volatility and stochastic interest rates.	21
5	Using the ADI method.	26
5.1	The first half step	27

5.2	The second half step	29
5.3	Boundary conditions.	31
6	Applying the ADI method in the case of stochastic volatility and stochastic interest rates (SVSI).	33
6.1	The first third of a step.	34
6.2	The second third of the step.	36
6.3	The final third of the step.	38
6.4	Boundary conditions.	40
7	An example of stochastic volatility and stochastic interest rates in options pricing.	43
8	An example of stochastic volatility and stochastic interest rates in fixed income derivatives pricing.	49
9	Data.	53
10	Results.	59
11	Conclusion.	67
A	A basic Black-Scholes framework for options pricing.	69

A.1	Continuously compounded returns and GBM.	71
A.2	The Black-Scholes-Merton (BSM) PDE.	71
A.3	Solving the PDE	72
A.4	Finite difference techniques.	73
B	Boundary conditions for the ADI method when solving the stochastic volatility PDE.	77
C	Boundary conditions for the ADI method when solving the SVSI PDE.	81
D	Characteristics of the bonds used as examples.	87

List of Figures

2.1	Graphs A and B plot the implied volatilities for Microsoft stock options as at Mar 11 5:14 PM EDT. The stock price at this time was \$29.28 per share. Graph A deals with the call options, whilst graph B works with put options.	7
7.1	One dimension, being either stock price (S), volatility (V), or interest rates (r) is fixed for each of the graphs. Interest rate=.05, volatility=.3, and stock price=50 for each of rows one, two and three respectively.	44
7.2	Volatility and interest rates are set equal to .3 and .05 respectively for each of these hedging sensitivities graphs.	45
8.1	Here are some graphs relating to price and sensitivities for convertible bonds.	50
9.1	Treasury bill rates obtained from the St. Louis Federal Reserve Economic Database (FRED).	54
10.1	Pricing errors for different models.	64

List of Tables

7.1	These hypothetical parameter values are used to produce the graphs that are to follow.	46
9.1	Estimated parameters for the interest rate process.	55
9.2	Estimated parameter values for the firm value processes.	58
10.1	Traded prices of bonds accompanied by their estimated values according to different pricing models.	60
10.2	The pricing error of each model as a percentage of the traded price.	61
10.3	The absolute pricing error of each model as a percentage of the traded price.	62
10.4	The second transaction involving Intel (INTC)	63
10.5	Mean squared error for each model	64
10.6	The third transaction for CSCO	65

Chapter 1

Introduction.

The aim of this paper is to compare the performance of different pricing models in valuing bonds with callable and convertible features. Additionally, we wish to provide a theoretical foundation and derivations of the models as we move through the paper. Much of the foundations for our approach to convertible bonds pricing, including optimal conditions for call and conversion, can be attributed to Ingersoll (1976) and Brennan and Schwartz (1977). These fundamental pricing conditions can then be built upon to arrive at more elaborate and numerically sophisticated models with the objective of more accurately pricing derivative securities.

The Black-Scholes (BS) model is the most commonly used model in valuing short term derivative instruments, such as equity derivatives, for example. As for longer term securities, such as convertible bonds, movements in volatility and interest rates are likely to have a compounding effect. Consequently, we conjecture that allowing for stochastic volatility and stochastic interest rates within the pricing of these longer term instruments is preferable. Additionally, given the much larger size of the fixed income derivatives markets when compared to other derivatives, it seems that the answer as to which pricing model is preferable carries significance.

As to the findings with regard to equity derivatives, Bakshi, Cao, and Chen (1997)

conclude that “taking stochastic volatility into account is of the first order importance in improving on the BS formula”, but “going from the SV to the SVSI does not necessarily improve the fit much further.” Firstly we shall look at pricing equity derivatives and convertible bonds using a more basic BS framework, then comparing this to the more complex SV and SVSI models later on in the paper. As for numerical pricing procedures, we concentrate on the use of the ADI finite difference method in order to estimate derivative values. Given the multiple variables that we wish to model, including firm value, volatility, and interest rates, we want a pricing procedure that is both accurate and computationally efficient. Whilst the ADI method is ideal for this situation, monte-carlo simulation is also an attractive approach to pricing convertible bonds. Indeed, in the case where the value of the option is path dependant, monte-carlo simulation is the ideal choice. To see examples of finite difference techniques used in the context of convertible bonds pricing, see Andersen and Buffum (2002). Alternatively, for a look into monte-carlo simulation, see Lvov, Yigitbasioglu, and Bachir (2004).

Chapter 2

Basic options pricing and practical application.

A European option gives the holder a right, but not an obligation, to purchase the underlying asset at a specified price upon maturity. American options, however, differ only in that they can be exercised at any time up to and including maturity. Additionally, these options could either be puts or calls. More specifically, put options give the option holder a right to sell the underlying asset whilst call options give the holder a right to buy. Note that the vast majority of exchange traded options are American in nature. In fact, all exchange traded stock and futures options in the US are American. Keep in mind that the name ‘American’ holds no geographical meaning; it is merely a name for options with these characteristics [Natenberg (1994)]. Closed form options pricing formulas are appropriate in the case of European style options, but pricing American options and dealing with the possibility of early exercise requires the use of numerical techniques: examples being binomial tree models, monte carlo simulation or a finite difference technique.

Note that, in the case of a stock option, the underlying stock may pay dividends at some point during the life of the option. In the case of fixed income derivatives, we not only need to concern ourselves with dividend payments, we also need to account

for any coupons that are paid. One way to handle dividends is to suppose that the stock value consists of two components: the present value of dividends over the option's life plus some additional value. This additional value is the component that is assumed to follow a geometric Brownian motion (GBM). In undertaking a finite difference approach, for example, we can implement the maturity value condition by stating that a call option's value is equal to $\max(S_t - K, 0)$ and a put option's value is $\max(K - S_T, 0)$, where S_T is the stock price at maturity and K is the strike price. Note that the present value of future dividends over the option's life will be zero at maturity. As we move backwards iteratively through each time step in the finite difference mesh, we need to ascertain whether early exercise is going to take place; to do so we take that part of the stock price that is following GBM and add the present value of future dividends, giving us the total stock value. The next step is then to compare the value of the option if exercised to its value if unexercised, and the larger of the two will be the option's value at that point. This process of backwards induction is continued iteratively for each step in the grid until we arrive at the option's value today.

A minor, yet still important, issue is how to go about measuring the option's time to maturity. For practical purposes, we can consider the option's life as being equal to the amount of days that the exchange is open from the present day until maturity. Therefore, weekends and public holidays during which the exchange is closed do not count towards the option's life. Note that the holiday schedule is identical on the NYSE and AMEX exchanges, and they are listed on the exchange's website. The 2009 holiday schedule, for example, is as follows:

- Jan 1 - New Year's Day.
- Jan 19 - Martin Luther King Jr's Birthday (Observed).
- Feb 16 - President's day.
- Apr 10 - Good Friday.
- May 25 - Memorial Day.
- Jul 2 - Early market close.

- Jul 3 - Independence day.
- Sep 7 -Labor day.
- Nov 26 - Thanksgiving day.
- Nov 27 - Early market close.
- Dec 24 - Early market close.
- Dec 25 - Christmas Day.

In total, there are approximately 252 trading days in the year such that $T \simeq \frac{\text{number of days in the options life}}{252}$. However, when it comes to the discounting of dividends we use a full 365 day year, weekends and public holidays inclusive. This is done because, regardless of whether the exchange is open or not, interest will accrue on deposits, and our discounting must reflect this opportunity cost.

Finally, lets turn our attention to some of the assumptions of the Black-Scholes PDE. Some of the main assumptions can be listed as:

- Constant Volatility.
- Constant and known interest rates.
- No commissions or transactions costs.
- The ability to adjust hedge positions continuously.
- Lognormally distributed returns.
- Efficient markets

Of course, we can specify processes for volatility and interest rates such that the first two of these assumptions are relaxed. This introduces it's own difficulties, as the PDE will get bigger and more complex to solve. Our main objective will be to deal with the incorporation of stochastic volatility and interest rates into the model.

When applying the basic Black-Scholes model in the real world, practitioners will often calculate implied volatilities and graph these across different maturities to create a volatility surface. Given that asset returns are not lognormally distributed, we often see that implied volatilities tend to ‘smile’. In other words, asset returns are typically more peaked and have fatter tails than a lognormal distribution would suggest, meaning that the probability of large price movements, either up or down, are likely to be larger than implied by our pricing model. Hence, options that are either significantly ITM or OTM are likely to be underpriced with the use of a lognormal distribution to model asset returns. Traders will be conscious of this fact and should adjust their pricing of options as a result. Consequently, this larger price will be reflected in a higher implied volatility. As an example, consider the graphs of implied volatilities for Microsoft options displayed in figure 2.1. Since implied volatility varies with strike price, it is clear that the assumption of constant volatility is a simplifying one. However, this basic pricing framework is considered to be good enough for the purposes of pricing relatively short maturity instruments. For an in depth discussion on how to price options using a basic Black Scholes framework, please see appendix A.

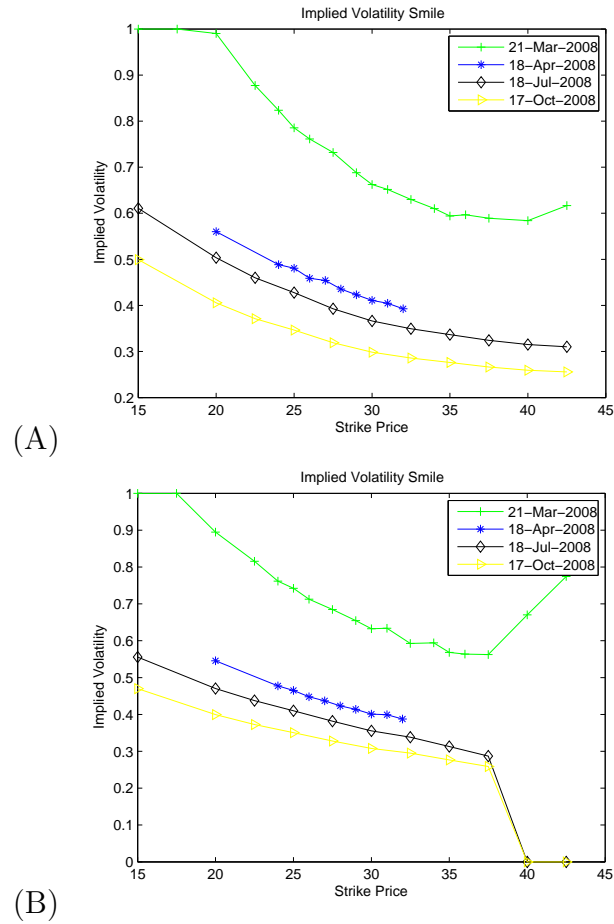


Figure 2.1: Graphs A and B plot the implied volatilities for Microsoft stock options as at Mar 11 5:14 PM EDT. The stock price at this time was \$29.28 per share. Graph A deals with the call options, whilst graph B works with put options.

Chapter 3

Foundations. The basic case.

Convertible bonds are significantly more complicated than stock options, for example; this is the case for a number of reasons. Firstly, stock options (ignoring LEAPS) typically have an expiration date spanning 9 months into the future or less. Convertible bonds, on the other hand, tend to involve maturity dates running far further into the future. Given the longer maturity of Convertible bonds, discounting becomes a far more significant consideration, and allowing for some means of modelling the evolution of interest rates is preferable.

Secondly, all stock options on the NYSE and AMEX exchanges are American in nature, meaning that they can be exercised at any time up to and including the maturity date; this option to purchase the underlying stock, however, only involves one potential claim (belonging to the option holder). As for convertible bonds, there is quite often a dual option scenario. Essentially, the holder of the convertible bond has the option to convert the bond into common stock, and the issuer may also have the option to call the bond at some point in the future. By definition, the call feature is simply a right, belonging to the bond issuer, to repurchase the bond at some percentage of par prior to maturity. Should the issuer decide to call the bond, then the holder has the right to either redeem the bond at the call price or convert.

Finally, the value of all derivatives is contingent on the value of some underlying asset. Consider the case of a stock option. Since the value of the underlying stock should fall by approximately the amount of the dividend upon the ex-dividend date, our numerical procedures designed to value such stock options need to take this into account. However, not only do we need to account for dividend payments when valuing fixed income derivatives, we also need to take account of when bond coupons are paid and incorporate the dual option scenario into our numerical procedures. The reason we need to concern ourselves with dividends and coupons is because they affect the value of the underlying asset, which in turn affects the value of the derivative.

The firm's right to call the bond is quite often restricted in some manner. For example, the bond may not be callable for a specified number of years, after which the call price may vary during certain time periods. As shown by Brennan and Schwartz (1977), the issuer of the convertible bond will want to undertake a call strategy so as to minimise the value of the bonds; after all, in doing so the issuer will be minimising their liabilities. The optimal call strategy for the issuer is then to call the bonds when the uncalled value of such bonds is equal to the call price. If the uncalled value of such bonds were below the call price, then calling the bonds would essentially amount to buying back something for more than it is worth. On the other hand, failing to call the bonds when their uncalled value is above the call price isn't consistent with the firm's objective to minimise the value of such bonds.

As for convertibility, the holders of convertible bonds will only convert their bonds if such an approach will maximise their value. Note that this objective of value maximisation for the bond holder is diametrically opposite to the intention of the issuer. Again, Brennan and Schwartz (1977) show that it is never optimal to convert an uncalled bond except at maturity or under circumstances immediately prior to the ex-dividend date or an unfavourable (from the bondholder's perspective) change in conversion terms. To see why this is the case, keep in mind that the bond can always be converted, suggesting that its value must be at least as high as the conversion value in order to satisfy no-arbitrage conditions.

Consider a firm whose capital structure consists entirely of common stock and one class of convertible securities. Similar to Brennan and Schwarz, we use the following definitions:

- $S(t)$ = aggregate market value at time t of the firm's outstanding securities including the convertible bonds.
- $f(S, t)$ = the market value at time t of one convertible bond with par value of \$1000.
- l = the number of convertible bonds outstanding.
- $n(t)$ = the number of shares of common stock into which each bond is convertible at time t .
- m = the number of shares of common stock outstanding before conversion takes place.
- I = the aggregate coupon payment on the outstanding convertible bonds at each periodic coupon date.
- $i = I/l$ = the periodic coupon payment per bond.
- $CP(t)$ = the price at which the bonds may be called for redemption at time t , including any accrued interest.
- $B(S, t)$ = the straight debt value of the bond; that is, the value of an otherwise identical bond with no conversion privilege.
- $D(S, t)$ = the aggregate dividend payment on the common stock at each dividend date.
- $VIC(S, t)$ = the value of the bond if it is called.

Should the bonds be converted, then the entire capital structure of the firm will consist of common stock. The Modigliani and Miller theorem would suggest that

firm value will be the same both before and after conversion. Noting that $S(t)/(m + n(t)l)$ is the value per share after conversion, we can write an expression for the conversion value per bond as:

$$\begin{aligned} C(S, t) &= n(t)[S(t)/(m + n(t)l)] \\ &\equiv z(t) \cdot S(t) \end{aligned} \tag{3.1}$$

By no-arbitrage arguments, the market value of the bond must be greater than or equal to the conversion value:

$$f(S, t) \geq C(S, t) \tag{3.2}$$

The VIC is the greater of the convertible and the callable value, given that the holder of the bond wants to maximise its value and gets to choose whether to redeem the bond at the call price or convert:

$$VIC(S, t) = \max[CP(t), C(S, t)] \tag{3.3}$$

All issuers of bonds will desire to minimise the value of their liabilities. Hence, the issuer will call the bond if such an action provides less value to the bond holder. Consequently, the market value of the bond can never be greater than the call price:

$$f(S, t) \leq CP(t) \tag{3.4}$$

The value of total bonds outstanding cannot exceed the firm value:

$$l \cdot f(S, t) \leq S(t) \tag{3.5}$$

The bonds are worthless if the firm goes bankrupt:

$$f(0, t) = 0 \tag{3.6}$$

We now turn our attention to the value of these convertible bonds at maturity. Should the conversion value of these bonds be greater than the par value at maturity, then the bond holders will seek to maximise their gains by converting. If, however, the par value is greater than the conversion value and the firm is not bankrupt (ie. $S(T) \geq l \cdot 1000$), then the bondholders will choose to receive the \$1000 face value. Finally, if the firm is bankrupt at maturity, meaning that the aggregate par value of bonds outstanding is greater than the firm value, then the value of the firm will be divided amongst the bondholders; given that there are l bonds outstanding, a holder of one convertible bond would receive S/l in this final case. These maturity value conditions can be summarised as:

$$f(S, t) = \begin{cases} z(T) \cdot S & \text{if } z(T) \cdot S > 1000 \\ 1000 & \text{if } 1000 \leq S \leq 1000/z(T) \\ S/l & \text{if } S < 1000 \cdot l \end{cases}$$

Note, however, that the bond may either be converted or called at some time prior to maturity. If the bond is currently callable, recall that the call price constraint is:

$$f(S, t) \leq CP(t)$$

If the bond is not currently callable, then the condition becomes:

$$\lim_{S \rightarrow \infty} f_s(S, t) = z(t) \tag{3.7}$$

This arises due to the fact that as firm value becomes exceptionally large, then it is virtually certain that the bonds are going to be converted into stock. Hence, each

bond could be considered as the right to buy a portion $z(t)$ of the firm, and the change in bond value for a small change in firm value is approximately equal to $z(t) \cdot \delta S$.

Finally, we need to consider what happens on dividend and coupon payment dates. Immediately after the ex-dividend date, the stock price will fall by approximately the amount of the dividend, thus resulting in a reduction in the conversion value of the bond. The bond holder is then faced with the choice of whether to convert before this happens or to simply hold on to the bond. Letting D be the aggregate dividend payment by the firm, we get:

$$f(S, t^-) = \max[f(S - D, t^+), z(t^-) \cdot S] \quad (3.8)$$

As for when coupons occur, let I be the aggregate amount of coupons paid by the firm and i be the amount of the coupon paid for each bond. Arbitrage conditions would dictate that, in the case where the bond is not currently callable, the pre-coupon bond value must be equal to the post-coupon bond value plus the amount of the coupon:

$$f(S, t^-) = f(S - I, t^+) + i \quad (3.9)$$

If the bond is currently callable, then the issuing firm will seek to minimise the aggregate value of the bonds:

$$f(S, t^-) = \min[f(S - I, t^+) + i, CP(t^-)] \quad (3.10)$$

3.1 Solving the basic PDE:

Assume that the underlying firm value follows Geometric Brownian Motion (GBM) such that:

$$dS = \mu S dt + \sigma S dz_t$$

Applying Ito's Lemma we arrive at:

$$df = \left[\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] dt + \sigma S \frac{\partial f}{\partial S} dz_t$$

Now create a portfolio by combining this bond with a portion $-\Delta = -\frac{\partial f}{\partial S}$ of the firm:

$$\begin{aligned} \Pi &= f - \frac{\partial f}{\partial S} S \\ d\Pi &= \left[\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] dt + \sigma S \frac{\partial f}{\partial S} dz_t - \frac{\partial f}{\partial S} [\mu S dt + \sigma S dz_t] \\ d\Pi &= \left[\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] dt \end{aligned}$$

Given that this portfolio is riskless, then it should yield the risk-free rate of interest:

$$\begin{aligned} d\Pi &= r\Pi dt = r \left[f - \frac{\partial f}{\partial S} S \right] dt = \left[\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] dt \\ r f - r S \frac{\partial f}{\partial S} &= \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \\ 0 &= \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - r f + \frac{\partial f}{\partial t} \end{aligned}$$

Instead of working with t , we'll work with τ , the time to maturity. Note that $\tau = (T - t)$. Differentiating, we get: $f_t = f_\tau \cdot \tau_t = -f_\tau$. Finally, we can apply this to the PDE such that:

$$0 = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - r f - \frac{\partial f}{\partial \tau}$$

This PDE can then be solved using standard numerical procedures. Since we are interested in solving the more complex PDE involving stochastic volatility and interest rates, the next section shall be dedicated to deriving the SVSI PDE.

Chapter 4

Adding more dimensions into the model.

Much of the derivation here can be found in Hull (2006). We include it here for completeness.

Firstly, let's look at Ito's lemma for a function of several variables. Suppose that $dx_i = a_i dt + b_i dz_i$, where dz_i is a Wiener process. This can be discretised as

$$\Delta x_i = a_i \Delta t + b_i \epsilon_i \sqrt{\Delta t}$$

Where ϵ is a standard normal random variable. Additionally, assume that $dz_i dz_j = \rho_{ij} dt$

We can then write Ito's Lemma for -

- One state variable as:

$$\Delta f = \frac{\partial f}{\partial x_1} (\Delta x_1) + \frac{\partial f}{\partial t} (\Delta t) + \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2} (\Delta x_1)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (\Delta t)^2 + \frac{\partial^2 f}{\partial x_1 \partial t} (\Delta x_1 \Delta t) + \dots$$

$$df = \frac{\partial f}{\partial x_1} (dx_1) + \frac{\partial f}{\partial t} (dt) + \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2} (dx_1)^2$$

- Two state variables as:

$$\begin{aligned}\Delta f &= \frac{\partial f}{\partial x_1}(\Delta x_1) + \frac{\partial f}{\partial x_2}(\Delta x_2) + \frac{\partial f}{\partial t}(\Delta t) + \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2}(\Delta x_1)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2}(\Delta x_2)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial t^2}(\Delta t)^2 \\ &\quad + \frac{\partial^2 f}{\partial x_1 \partial x_2}(\Delta x_1 \Delta x_2) + \frac{\partial^2 f}{\partial x_1 \partial t}(\Delta x_1 \Delta t) + \frac{\partial^2 f}{\partial x_2 \partial t}(\Delta x_2 \Delta t) + \dots\end{aligned}$$

$$df = \frac{\partial f}{\partial x_1}(dx_1) + \frac{\partial f}{\partial x_2}(dx_2) + \frac{\partial f}{\partial t}(dt) + \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2}(dx_1)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2}(dx_2)^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(dx_1 dx_2)$$

- In general as:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(dx_i) + \frac{\partial f}{\partial t}(dt) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(dx_i dx_j)$$

Note that:

$$dx_i = a_i dt + b_i dz_i$$

Therefore, we can write

$$\begin{aligned}dx_i dx_j &= (a_i dt + b_i dz_i) \cdot (a_j dt + b_j dz_j) \\ &= b_i b_j (dz_i dz_j)\end{aligned}$$

Recalling that $dz_i dz_j = \rho_{ij} dt$, then we can substitute:

$$dx_i dx_j = b_i b_j \rho_{ij} dt.$$

Now substitute this into the general form of Ito's Lemma:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(dx_i) + \frac{\partial f}{\partial t}(dt) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij} dt$$

Finally, substitute for $dx_i = a_i dt + b_i dz_i$:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a_i dt + b_i dz_i) + \frac{\partial f}{\partial t}(dt) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij} dt$$

Now group all terms involving dt :

$$df = \left[\sum_{i=1}^n \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij} \right] dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} b_i dz_i$$

4.1 Multiple state variables.

Theorem 4.1.1 *Suppose that we have several state variables $\theta_1, \theta_2, \dots, \theta_n$ such that:*

$$d\theta_i = m_i \theta_i dt + s_i \theta_i dz_i.$$

Where dz_i is a Weiner process. Suppose that there are n derivatives, each of whose price is f_j , and that their values follow:

$$df_j = \eta_j f_j dt + \sum_{i=1}^n \sigma_{ij} f_j dz_i$$

Given this information, we can then write:

$$\eta_j - r = \sum_{i=1}^n \lambda_i \sigma_{ij}$$

Proof Define k_j as the amount of security j in some portfolio of derivatives such that:

$$\Pi = \sum_{j=1}^n k_j f_j$$

$$d\Pi = \sum_{j=1}^n k_j \left[\eta_j f_j dt + \sum_{i=1}^n \sigma_{ij} f_j dz_i \right]$$

Now choose the k_j s so as to eliminate the stochastic component in the above equation. To achieve this, set $\sum_{j=1}^n k_j \sigma_{ij} f_j = 0$ for all $i = 1 \dots n$. We are then left

with:

$$d\Pi = \sum_{j=1}^n k_j \eta_j f_j dt$$

The cost of setting up this portfolio is $\sum_{j=1}^n k_j f_j$. This portfolio is riskless, so, by no-arbitrage arguments, it should yield the risk-free interest rate. Therefore:

$$\begin{aligned} d\Pi &= r\Pi dt \\ \sum_{j=1}^n k_j \eta_j f_j dt &= r \sum_{j=1}^n k_j f_j dt \\ \sum_{j=1}^n k_j \eta_j f_j &= r \sum_{j=1}^n k_j f_j \\ \sum_{j=1}^n k_j f_j (\eta_j - r) &= 0 \end{aligned}$$

Note that $\sum_{j=1}^n k_j \sigma_{ij} f_j = 0$ for $i = 1 \dots n$ and $\sum_{j=1}^n k_j f_j (\eta_j - r) = 0$. These $n + 1$ equations in $k_j f_j$ can only be consistent if the last equation is a linear combination of the others: $\eta_j - r = \sum_{i=1}^n \lambda_i \sigma_{ij}$. ■

4.2 Application to options pricing with stochastic volatility.

Proposition 1 *Suppose that:*

$$\begin{aligned} dS &= \mu S dt + \sqrt{V} S dz_1 \\ dV &= (\theta_v - \kappa_v V) dt + \sigma_v \sqrt{V} dz_2 \\ E[dz_1 \cdot dz_2] &= \rho dt \end{aligned}$$

Then the price of any derivative whose payoff depends on S and V obeys the PDE:

$$rf = rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V} (\theta_v - \kappa_v V - \lambda_2 \sigma_v \sqrt{V}) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} V \sigma_v^2 + \frac{\partial^2 f}{\partial S \partial V} S \sigma_v \rho_{sv}$$

Where λ_2 is a function of S , V and t .

Proof Applying Ito's lemma, we get:

$$df = \left[\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial V} (\theta_v - \kappa_v V) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} V \sigma_v^2 + \frac{\partial^2 f}{\partial S \partial V} S \sigma_v \rho_{sv} \right] dt \\ + \frac{\partial f}{\partial S} \sqrt{V} S dz_1 + \frac{\partial f}{\partial V} \sigma_v \sqrt{V} dz_2$$

From Theorem 4.1.1:

- $df = \eta f dt + \sum_{i=1}^n \sigma_i f dz_i$
- $\eta - r = \sum_{i=1}^n \lambda_i \sigma_i$

In this particular case with stochastic volatility:

$$\eta f = \left[\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial V} (\theta_v - \kappa_v V) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} V \sigma_v^2 + \frac{\partial^2 f}{\partial S \partial V} S \sigma_v \rho_{sv} \right] \\ = rf + \sum_{i=1}^n \lambda_i \sigma_i f \\ = rf + \lambda_1 \frac{\partial f}{\partial S} \sqrt{V} S + \lambda_2 \frac{\partial f}{\partial V} \sigma_v \sqrt{V}$$

Consider the process followed by the stock price:

$$dS = \mu S dt + \sqrt{V} S dz_1 + 0 \cdot dz_2$$

Therefore, $\mu = r + \lambda_1 \sqrt{V}$, such that $\mu - r = \lambda_1 \sqrt{V}$. With this in mind, we can write:

$$rf + \lambda_1 \frac{\partial f}{\partial S} \sqrt{V} S + \lambda_2 \frac{\partial f}{\partial V} \sigma_v \sqrt{V}$$

$$\begin{aligned}
&= rf + \frac{\partial f}{\partial S}S(\mu - r) + \lambda_2 \frac{\partial f}{\partial V}\sigma_v\sqrt{V} \\
&= \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial V}(\theta_v - \kappa_v V) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}VS^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}V\sigma_v^2 + \frac{\partial^2 f}{\partial S\partial V}S\sigma_v\rho_{sv} \right]
\end{aligned}$$

Therefore:

$$\begin{aligned}
&rf + \lambda_2 \frac{\partial f}{\partial V}\sigma_v\sqrt{V} \\
&= \left[rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V}(\theta_v - \kappa_v V) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}VS^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}V\sigma_v^2 + \frac{\partial^2 f}{\partial S\partial V}S\sigma_v\rho_{sv} \right]
\end{aligned}$$

Now rearrange terms such that only rf is on the left hand side of the equation:

$$rf = rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V}(\theta_v - \kappa_v V - \lambda_2 \sigma_v \sqrt{V}) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}VS^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}V\sigma_v^2 + \frac{\partial^2 f}{\partial S\partial V}S\sigma_v\rho_{sv} \quad \blacksquare$$

Corollary 1 Suppose that the conditions of proposition 1 hold in addition to $\lambda_2 \sigma_v \sqrt{V} \equiv \lambda_3 V$. Additionally, define $\kappa_{v2} \equiv \kappa_v + \lambda_3$. The PDE will then become:

$$rf = rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V}(\theta_v - \kappa_{v2} V) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}VS^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}V\sigma_v^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}V\sigma_v^2 + \frac{\partial^2 f}{\partial S\partial V}S\sigma_v\rho_{sv}.$$

Proof Assume that $\lambda_2 \sigma_v \sqrt{V} \equiv \lambda_3 V$, where λ_3 is a constant. Therefore, $\theta_v - \kappa_v V - \lambda_2 \sigma_v \sqrt{V} = \theta_v - \kappa_v V - \lambda_3 V = \theta_v - (\kappa_v + \lambda_3)V \equiv \theta_v - \kappa_{v2} V$. Note that we have defined $\kappa_{v2} \equiv \kappa_v + \lambda_3$. Finally, the PDE becomes:

$$rf = rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V}(\theta_v - \kappa_{v2} V) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}VS^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}V\sigma_v^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}V\sigma_v^2 + \frac{\partial^2 f}{\partial S\partial V}S\sigma_v\rho_{sv}. \quad \blacksquare$$

4.3 Application to options pricing with stochastic volatility and stochastic interest rates.

Proposition 2 *Consistent with Bakshi, Cao and Chen (1997), suppose that the stock price, volatility and interest rate follow the following processes:*

$$\begin{aligned}
 dS &= \mu S dt + \sqrt{V} S dz_1 \\
 dV &= [\theta_v - \kappa_v V] dt + \sigma_v \sqrt{V} dz_2 \\
 dr &= [\theta_r - \kappa_r r] dt + \sigma_r \sqrt{r} dz_3 \\
 E(dz_1 dz_2) &= \rho dt \\
 E(dz_1 dz_3) &= E(dz_2 dz_3) = 0
 \end{aligned}$$

This will then result in the following PDE:

$$\begin{aligned}
 rf &= rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V} [(\theta_v - \kappa_v V) - \lambda_2 \sigma_v \sqrt{V}] + \frac{\partial f}{\partial r} [(\theta_r - \kappa_r r) - \lambda_3 \sigma_r \sqrt{r}] \\
 &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V + \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} V S \sigma_v \rho_{sv}
 \end{aligned}$$

Where λ_2 and λ_3 are functions of S , V and t . This PDE will be satisfied by any derivative price.

Proof Firstly, note that interest rates are uncorrelated with volatility and the stock price. Additionally, suppose that $f(S, V, r, t)$ is a derivative price. Applying Ito's lemma, we then arrive at:

$$\begin{aligned}
df &= \left[\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial V} [\theta_v - \kappa_v V] + \frac{\partial f}{\partial r} [\theta_r - \kappa_r r] + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} \rho V \sigma_v S \right] dt \\
&\quad + \frac{\partial f}{\partial S} \sqrt{V} S dz_1 + \frac{\partial f}{\partial V} \sigma_v \sqrt{V} dz_2 + \frac{\partial f}{\partial r} \sigma_r \sqrt{r} dz_3.
\end{aligned}$$

Again, from Theorem 4.1.1:

- $df = \eta f dt + \sum_{i=1}^n \sigma_i f dz_i$
- $\eta - r = \sum_{i=1}^n \lambda_i \sigma_i$

In this particular case:

$$\begin{aligned}
\eta f &= \left[\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial V} [\theta_v - \kappa_v V] + \frac{\partial f}{\partial r} [\theta_r - \kappa_r r] + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} \rho V \sigma_v S \right] \\
&= r f + \sum_{i=1}^n \lambda_i \sigma_i f \\
&= r f + \lambda_1 \left[\frac{\partial f}{\partial S} \sqrt{V} S \right] + \lambda_2 \left[\frac{\partial f}{\partial V} \sigma_v \sqrt{V} \right] + \lambda_3 \left[\frac{\partial f}{\partial r} \sigma_r \sqrt{r} \right]
\end{aligned}$$

The process followed by the stock price is:

$$dS = \mu S dt + \sqrt{V} S dz_1 + 0 \cdot dz_2 + 0 \cdot dz_3.$$

Therefore, $\mu = r + \lambda_1 \sqrt{V}$, such that $\mu - r = \lambda_1 \sqrt{V}$. Consequently, we can write:

$$\begin{aligned}
& rf + \lambda_1 \left[\frac{\partial f}{\partial S} \sqrt{V} S \right] + \lambda_2 \left[\frac{\partial f}{\partial V} \sigma_v \sqrt{V} \right] + \lambda_3 \left[\frac{\partial f}{\partial r} \sigma_r \sqrt{r} \right] \\
&= rf + (\mu - r) \frac{\partial f}{\partial S} S + \lambda_2 \left[\frac{\partial f}{\partial V} \sigma_v \sqrt{V} \right] + \lambda_3 \left[\frac{\partial f}{\partial r} \sigma_r \sqrt{r} \right] \\
&= \left[\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial V} [\theta_v - \kappa_v V] + \frac{\partial f}{\partial r} [\theta_r - \kappa_r r] + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} \rho V \sigma_v S \right]
\end{aligned}$$

Therefore:

$$\begin{aligned}
& rf - rS \frac{\partial f}{\partial S} + \lambda_2 \left[\frac{\partial f}{\partial V} \sigma_v \sqrt{V} \right] + \lambda_3 \left[\frac{\partial f}{\partial r} \sigma_r \sqrt{r} \right] \\
&= \left[\frac{\partial f}{\partial V} [\theta_v - \kappa_v V] + \frac{\partial f}{\partial r} [\theta_r - \kappa_r r] + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} \rho V \sigma_v S \right]
\end{aligned}$$

Now rearrange this equation such that rf is on the left hand side:

$$\begin{aligned}
rf &= rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V} [(\theta_v - \kappa_v V) - \lambda_2 \sigma_v \sqrt{V}] + \frac{\partial f}{\partial r} [(\theta_r - \kappa_r r) - \lambda_3 \sigma_r \sqrt{r}] \\
&\quad + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} V S \sigma_v \rho_{sv} \quad \blacksquare
\end{aligned}$$

Corollary 2 *Suppose that the conditions of proposition 2 hold in addition to:*

- $\lambda_2 \sigma_v \sqrt{V} \equiv \lambda_4 V$
- $\lambda_3 \sigma_r \sqrt{r} \equiv \lambda_5 r$

Where λ_4 and λ_5 are constants.

Additionally, we shall make these definitions:

- $\kappa_{v3} \equiv \kappa_v + \lambda_4$.
- $\kappa_{r3} \equiv \kappa_r + \lambda_5$.

The PDE will then become:

$$rf = rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V} [\theta_v - \kappa_{v3} V] + \frac{\partial f}{\partial r} [\theta_r - \kappa_{r3} r] - \frac{\partial f}{\partial \tau} \\ + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} \rho V \sigma_v S$$

Proof Assume that $\lambda_2 \sigma_v \sqrt{v} \equiv \lambda_4 V$, such that $(\theta_v - \kappa_v V) - \lambda_2 \sigma_v \sqrt{v} = (\theta_v - \kappa_v V) - \lambda_4 V \equiv (\theta_v - \kappa_{v3} V)$, where $\kappa_{v3} \equiv (\kappa_v + \lambda_4)$.

Additionally, assume that $\lambda_3 \sigma_r \sqrt{r} \equiv \lambda_5 r$. Therefore, $(\theta_r - \kappa_r r) - \lambda_3 \sigma_r \sqrt{r} = (\theta_r - \kappa_r r) - \lambda_5 r \equiv (\theta_r - \kappa_{r3} r)$, where $\kappa_{r3} \equiv (\kappa_r + \lambda_5)$.

The PDE will then become:

$$rf = rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V} [\theta_v - \kappa_{v3} V] + \frac{\partial f}{\partial r} [\theta_r - \kappa_{r3} r] + \frac{\partial f}{\partial t} \\ + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} \rho V \sigma_v S$$

Recalling that $\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial \tau}$, we finally arrive at:

$$rf = rS \frac{\partial f}{\partial S} + \frac{\partial f}{\partial V} [\theta_v - \kappa_{v3} V] + \frac{\partial f}{\partial r} [\theta_r - \kappa_{r3} r] - \frac{\partial f}{\partial \tau} \\ + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} V S^2 + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma_v^2 V + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} \sigma_r^2 r + \frac{\partial^2 f}{\partial S \partial V} \rho V \sigma_v S \quad \blacksquare$$

When we come to the task of estimation, keep in mind that we shall be assuming $\kappa_{v3} = \kappa_v$. In other words, $\lambda_4 = 0$. Additionally, we shall be supposing that $\lambda_5 = 0$ such that $\kappa_{r3} = \kappa_r$. Essentially, we shall be taking the common practice approach of treating the real-world and risk-neutral probabilities as though they were equal.

Chapter 5

Using the ADI method.

Now that we have relaxed the assumption of constant volatility and obtained an appropriate PDE, we are still faced with the task of solving this PDE. Fortunately, the Alternating Direction Implicit (ADI) method can be a useful tool for this purpose. For a basic example of applying the ADI method in solving the heat equation see Brandimarte (2006). At its core, the ADI method introduces an intermediary time step into the solution, thus reducing a potential multi-dimensional problem into a series of one dimensional problems. To see how this works, perhaps it is best to see an example¹. Keeping in mind that we are dealing with an intermediary time step, we can use the ADI method to solve the PDE derived for stochastic volatility in the previous section.

Our PDE is as follows:

$$rf = rS \frac{\partial f}{\partial S} + (\theta_v - \kappa_v 2V) \frac{\partial f}{\partial V} + \frac{\partial f}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} V \sigma_v^2 + \frac{\partial^2 f}{\partial S \partial V} V S \sigma_v \rho_{sv}$$

Note that, in the case of a stock option or callable/convertible bond, we know the value at maturity. Hence we can calculate the value at maturity of our instrument

¹For an explanation of how to solve similar problems using Monte-Carlo simulation, see Bakshi et al. (1997) or Hull and White (1987)

and work backwards in time until we obtain the theoretical value today. With this in mind, and with the purpose of avoiding confusion, we shall work with time to maturity (τ) instead of the current period t . Recalling that $\tau = (T - t)$, we can then state $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = \frac{\partial f}{\partial \tau} \cdot -1 = -\frac{\partial f}{\partial \tau}$. Finally, our new PDE when working with τ is:

$$rf = rS \frac{\partial f}{\partial S} + (\theta_v - \kappa_{v2}V) \frac{\partial f}{\partial V} - \frac{\partial f}{\partial \tau} + \frac{1}{2}VS^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} V\sigma_v^2 + \frac{\partial^2 f}{\partial S \partial V} VS\sigma_v\rho_{sv}$$

5.1 The first half step

We shall use the superscript i and the subscripts j and k to denote the τ , stock price and volatility dimensions respectively. Our finite difference approximations for the first half step will then be:

$$\begin{aligned} \frac{\partial f}{\partial \tau} &\simeq \frac{f_{j,k}^{i+\frac{1}{2}} - f_{j,k}^i}{(\Delta\tau)/2} \\ \frac{\partial f}{\partial S} &\simeq \frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{2\Delta S} \\ \frac{\partial^2 f}{\partial S^2} &\simeq \frac{f_{j+1,k}^{i+\frac{1}{2}} - 2f_{j,k}^{i+\frac{1}{2}} + f_{j-1,k}^{i+\frac{1}{2}}}{\Delta S^2} \\ \frac{\partial f}{\partial V} &\simeq \frac{f_{j,k+1}^i - f_{j,k-1}^i}{2\Delta V} \\ \frac{\partial^2 f}{\partial V^2} &\simeq \frac{f_{j,k+1}^i - 2f_{j,k}^i + f_{j,k-1}^i}{\Delta V^2} \\ \frac{\partial^2 f}{\partial S \partial V} &\simeq \frac{f_{j+1,k+1}^i - f_{j+1,k-1}^i - f_{j-1,k+1}^i + f_{j-1,k-1}^i}{4\Delta V \Delta S} \end{aligned}$$

Now use these as approximations to the partial derivatives in our PDE to obtain:

$$\begin{aligned}
rf_{j,k}^{i+\frac{1}{2}} &= rS_j \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{2\Delta S} \right] + (\theta_v - \kappa_{v2}V_k) \left[\frac{f_{j,k+1}^i - f_{j,k-1}^i}{2\Delta V} \right] - \left[\frac{f_{j,k}^{i+\frac{1}{2}} - f_{j,k}^i}{(\Delta\tau)/2} \right] \\
&+ \frac{1}{2}V_kS_j^2 \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - 2f_{j,k}^{i+\frac{1}{2}} + f_{j-1,k}^{i+\frac{1}{2}}}{\Delta S^2} \right] + \frac{1}{2}V_k\sigma_v^2 \left[\frac{f_{j,k+1}^i - 2f_{j,k}^i + f_{j,k-1}^i}{\Delta V^2} \right] \\
&+ V_kS_j\sigma_v\rho_{sv} \left[\frac{f_{j+1,k+1}^i - f_{j+1,k-1}^i - f_{j-1,k+1}^i + f_{j-1,k-1}^i}{4\Delta V\Delta S} \right]
\end{aligned}$$

Now rearrange this equation such that all the terms involving τ step $i + \frac{1}{2}$ are on the left hand side:

$$\begin{aligned}
&f_{j-1,k}^{i+\frac{1}{2}} \left[\frac{rS_j}{2(\Delta S)} - \frac{V_kS_j^2}{2(\Delta S)^2} \right] + f_{j,k}^{i+\frac{1}{2}} \left[r + \frac{1}{(\Delta\tau)/2} + \frac{V_kS_j^2}{(\Delta S)^2} \right] \\
&+ f_{j+1,k}^{i+\frac{1}{2}} \left[\frac{-rS_j}{2(\Delta S)} - \frac{V_kS_j^2}{2(\Delta S)^2} \right] \\
&= f_{j,k-1}^i \left[\frac{-(\theta_v - \kappa_{v2}V_k)}{2(\Delta V)} + \frac{V_k\sigma_v^2}{2(\Delta V)^2} \right] + f_{j,k}^i \left[\frac{1}{(\Delta\tau)/2} - \frac{V_k\sigma_v^2}{(\Delta V)^2} \right] \\
&+ f_{j,k+1}^i \left[\frac{(\theta_v - \kappa_{v2}V_k)}{2(\Delta V)} + \frac{V_k\sigma_v^2}{2(\Delta V)^2} \right] \\
&+ V_kS_j\sigma_v\rho_{sv} \left[\frac{f_{j+1,k+1}^i - f_{j+1,k-1}^i - f_{j-1,k+1}^i + f_{j-1,k-1}^i}{4(\Delta V)(\Delta S)} \right]
\end{aligned} \tag{5.1}$$

Notice that everything on the right hand side of the equation involves time step i , meaning that everything is known on the right hand side. However, all the terms on the left hand side of the equation involve time step $i + \frac{1}{2}$, the values which we currently don't know. Keep in mind that the time and volatility dimensions, being $i + \frac{1}{2}$ and k respectively, are fixed. Consequently, only the stock price dimension is varying and the problem has effectively become one-dimensional. Obtaining values for the intermediary time step will essentially involve solving this equation for each level of k . Mathematically, this will involve solving a set of $J \times J$ tri-diagonal

matrices as we move through each level of volatility in the finite difference grid $(V_0, \dots, V_k, \dots, V_K)$. This is far more efficient than solving one $JK \times JK$ system of equations.

5.2 The second half step

We are now faced with the task of moving from the auxiliary time step $(i + \frac{1}{2})$ to time step $(i + 1)$. Again, we need to obtain finite difference approximations to the partial derivatives in our PDE. However, this time we shall do the stock price approximations at time step $(i + \frac{1}{2})$ and the volatility approximations at time step $(i + 1)$. As a result, the finite difference approximations will be as follows:

$$\begin{aligned} \frac{\partial f}{\partial \tau} &\simeq \frac{f_{j,k}^{i+1} - f_{j,k}^{i+\frac{1}{2}}}{(\Delta\tau)/2} \\ \frac{\partial f}{\partial S} &\simeq \frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{2\Delta S} \\ \frac{\partial^2 f}{\partial S^2} &\simeq \frac{f_{j+1,k}^{i+\frac{1}{2}} - 2f_{j,k}^{i+\frac{1}{2}} + f_{j-1,k}^{i+\frac{1}{2}}}{(\Delta S)^2} \\ \frac{\partial f}{\partial V} &\simeq \frac{f_{j,k+1}^{i+1} - f_{j,k-1}^{i+1}}{2\Delta V} \\ \frac{\partial^2 f}{\partial V^2} &\simeq \frac{f_{j,k+1}^{i+1} - 2f_{j,k}^{i+1} + f_{j,k-1}^{i+1}}{(\Delta V)^2} \\ \frac{\partial^2 f}{\partial S \partial V} &\simeq \frac{f_{j+1,k+1}^{i+\frac{1}{2}} - f_{j+1,k-1}^{i+\frac{1}{2}} - f_{j-1,k+1}^{i+\frac{1}{2}} + f_{j-1,k-1}^{i+\frac{1}{2}}}{4\Delta V \Delta S} \end{aligned}$$

Recalling that the PDE is:

$$rf = rS \frac{\partial f}{\partial S} + (\theta_v - \kappa_v V) \frac{\partial f}{\partial V} + \frac{\partial f}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} V \sigma_v^2 + \frac{\partial^2 f}{\partial S \partial V} V S \sigma_v \rho_{sv}$$

We can then input the finite difference approximations into the PDE to obtain:

$$\begin{aligned}
rf_{j,k}^{i+\frac{1}{2}} &= rS_j \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{2\Delta S} \right] + (\theta_v - \kappa_v 2V) \left[\frac{f_{j,k+1}^{i+1} - f_{j,k-1}^{i+1}}{2\Delta V} \right] - \left[\frac{f_{j,k}^{i+1} - f_{j,k}^{i+\frac{1}{2}}}{(\Delta\tau)/2} \right] \\
&+ \frac{1}{2}V_k S_j^2 \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - 2f_{j,k}^{i+\frac{1}{2}} + f_{j-1,k}^{i+\frac{1}{2}}}{(\Delta S)^2} \right] + \frac{1}{2}V_k \sigma_v^2 \left[\frac{f_{j,k+1}^{i+1} - 2f_{j,k}^{i+1} + f_{j,k-1}^{i+1}}{(\Delta V)^2} \right] \\
&+ V_k S_j \sigma_v \rho_{sv} \left[\frac{f_{j+1,k+1}^{i+\frac{1}{2}} - f_{j+1,k-1}^{i+\frac{1}{2}} - f_{j-1,k+1}^{i+\frac{1}{2}} + f_{j-1,k-1}^{i+\frac{1}{2}}}{4\Delta V \Delta S} \right]
\end{aligned}$$

This can be rearranged such that all of the terms involving τ step $(i+1)$ are on the left hand side of the equation:

$$\begin{aligned}
&f_{j,k+1}^{i+1} \left[\frac{-(\theta_v - \kappa_v 2V_k)}{2(\Delta V)} - \frac{V_k \sigma_v^2}{2(\Delta V)^2} \right] + f_{j,k}^{i+1} \left[\frac{1}{(\Delta\tau/2)} + \frac{V_k \sigma_v^2}{(\Delta V)^2} \right] \\
&+ f_{j,k-1}^{i+1} \left[\frac{(\theta_v - \kappa_v 2V_k)}{2(\Delta V)} - \frac{V_k \sigma_v^2}{2(\Delta V)^2} \right] \\
&= f_{j+1,k}^{i+\frac{1}{2}} \left[\frac{rS_j}{2(\Delta S)} + \frac{V_k S_j^2}{2(\Delta S)^2} \right] + f_{j,k}^{i+\frac{1}{2}} \left[-r + \frac{1}{(\Delta\tau/2)} - \frac{V_k S_j^2}{(\Delta S)^2} \right] \\
&+ f_{j-1,k}^{i+\frac{1}{2}} \left[\frac{-rS_j}{2(\Delta S)} + \frac{V_k S_j^2}{2(\Delta S)^2} \right] \\
&+ V_k S_j \sigma_v \rho_{sv} \left[\frac{f_{j+1,k+1}^{i+\frac{1}{2}} - f_{j+1,k-1}^{i+\frac{1}{2}} - f_{j-1,k+1}^{i+\frac{1}{2}} + f_{j-1,k-1}^{i+\frac{1}{2}}}{4(\Delta V)(\Delta S)} \right]
\end{aligned} \tag{5.2}$$

Again, notice that all terms on the right hand side of the above expression are known after the first half-step. As for the terms on the left hand side of the equation, these terms all involve a fixed dimension for τ and the stock price, being $(i+1)$ and j respectively. In other words, for the unknown components of the equation, being the left hand side, only the volatility dimension is varying. Hence we once again see that the problem has been made one-dimensional. Implementing this equation and moving our solution from the intermediary time step to τ level $(i+1)$ in the finite difference grid will then involve solving a set of tri-diagonal matrices for each level of the stock price $(S_0, \dots, S_j, \dots, S_J)$.

5.3 Boundary conditions.

We have already shown that, provided we are not on a boundary in the finite difference mesh, the equations governing the first and second half-steps are (5.1) and (5.2).

For the sake of simplicity, I shall write these equations more concisely as:

The first half-step:

$$f_{j-1,k}^{i+\frac{1}{2}}\xi_1 + f_{j,k}^{i+\frac{1}{2}}\xi_2 + f_{j+1,k}^{i+\frac{1}{2}}\xi_3 = f_{j,k-1}^i\xi_4 + f_{j,k}^i\xi_5 + f_{j,k+1}^i\xi_6 + V_k S_j \sigma_v \rho_{sv} \xi_7$$

The second half-step:

$$f_{j,k+1}^{i+1}\xi_8 + f_{j,k}^{i+1}\xi_9 + f_{j,k-1}^{i+1}\xi_{10} = f_{j+1,k}^{i+\frac{1}{2}}\xi_{11} + f_{j,k}^{i+\frac{1}{2}}\xi_{12} + f_{j-1,k}^{i+\frac{1}{2}}\xi_{13} + V_k S_j \sigma_v \rho_{sv} \xi_{14}$$

Finally, consider possible boundaries for the finite difference mesh. There are several possibilities to consider in this example. Firstly, we know what the value of a stock option will be at maturity for a given stock and strike price, and we wish to work through the grid so as to find the value of the derivative today (at $\tau = T$). Hence the time dimension boundaries won't have an impact on the above equations. However, there is still the matter of the stock price and volatility dimensions. At a given iteration in our finite difference mesh we could either be operating on a stock price boundary, a volatility dimension boundary or a combination of both.

Volatility:

- Upper boundary (V_K)
- Not on a boundary.
- Lower boundary (V_0)

Stock Price:

- Upper boundary (S_J)
- Not on a boundary.
- Lower boundary (S_0)

Considering the case of a stock option, $\frac{\partial f}{\partial S}$ is likely to be relatively constant for very high and low stock prices (i.e. at the boundaries). As a result, we could safely assume that $\frac{\partial^2 f}{\partial S^2}$ would be zero at S_J and S_0 . Also note that if $\frac{\partial f}{\partial S}$ is relatively constant, then we can also assume $\frac{\partial^2 f}{\partial S \partial V}$ is equal to zero. Additionally, whether we are at the bounds of S for our finite difference mesh will influence how we define the finite difference approximations. For example, if we are working at the upper boundary for the stock price, being S_J , then we will have to redefine $\frac{\partial f}{\partial S}$ such that our approximation does not refer to points outside the grid (i.e. S_{J+1}). The same is also true for the lower boundary where we would have to redefine $\frac{\partial f}{\partial S}$ such that it does not incorporate a point S_{-1} in the stock price dimension, as such a point does not exist.

Turning our attention to the upper and lower boundaries of the volatility dimension, as volatility gets either very large or very small, being the boundaries of the volatility dimension, we shall assume that $\frac{\partial f}{\partial V}$ is relatively constant. Effectively, this will also result in $\frac{\partial^2 f}{\partial V^2}$ and $\frac{\partial^2 f}{\partial S \partial V}$ being zero at the volatility boundaries. This makes intuitive sense, as regardless of how large or small the volatility becomes, no arbitrage arguments allow us to set upper and lower boundaries for the option's value. For example, regardless of how high volatility becomes, a European call option can never trade at a higher value than the underlying stock ($c \leq S_0$); conversely, the lower price boundary for a European call option in the case of no dividends is $S_0 - Ke^{-rT}$.²

For a full discussion of how the numerical procedure alters at the boundaries of this PDE, please refer to appendix B.

²For more of a discussion of option price boundaries, see Hull (2006) Chap. 9.

Chapter 6

Applying the ADI method in the case of stochastic volatility and stochastic interest rates (SVSI).

We have previously relaxed the assumption of constant volatility, and now we are going to do the same for interest rates. Note that the ADI approach will be similar, except that we now need two intermediary time steps and, of course, the equations will be different. The PDE when dealing with stochastic volatility and stochastic interest rates has been derived as:

$$rf = \frac{1}{2}VS^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \rho\sigma_v VS\frac{\partial^2 f}{\partial S\partial V} + \frac{1}{2}\sigma_v^2 V\frac{\partial^2 f}{\partial V^2} + [\theta_v - \kappa_{v3}V]\frac{\partial f}{\partial V} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 f}{\partial r^2} + [\theta_r - \kappa_{r3}r]\frac{\partial f}{\partial r} - \frac{\partial f}{\partial \tau}$$

Our numerical procedure using the ADI method will work through each time step iteratively. Given that there are two intermediary time steps, there will be a total of three iterations that will have to be made in order to progress between each

time step within the finite difference mesh. In keeping with the previous notation, we shall express the time dimension as i , the stock dimension as j , the volatility dimension as k and the interest rate dimension as l . Hence, we shall express the first intermediary time step as $(i + \frac{1}{3})$, the second intermediary time step as $(i + \frac{2}{3})$ and the final step in our grid as $(i + 1)$.

6.1 The first third of a step.

We shall define the finite difference approximations (FDAs) as follows:

$$\begin{aligned} \frac{\partial f}{\partial \tau} &\simeq \frac{f_{j,k,l}^{i+\frac{1}{3}} - f_{j,k,l}^i}{(\Delta\tau/3)} \\ \frac{\partial f}{\partial S} &\simeq \frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \\ \frac{\partial^2 f}{\partial S^2} &\simeq \frac{f_{j+1,k,l}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j-1,k,l}^{i+\frac{1}{3}}}{(\Delta S)^2} \\ \frac{\partial f}{\partial V} &\simeq \frac{f_{j,k+1,l}^i - f_{j,k-1,l}^i}{2\Delta V} \\ \frac{\partial^2 f}{\partial V^2} &\simeq \frac{f_{j,k+1,l}^i - 2f_{j,k,l}^i + f_{j,k-1,l}^i}{(\Delta V)^2} \\ \frac{\partial f}{\partial r} &\simeq \frac{f_{j,k,l+1}^i - f_{j,k,l-1}^i}{2\Delta r} \\ \frac{\partial^2 f}{\partial r^2} &\simeq \frac{f_{j,k,l+1}^i - 2f_{j,k,l}^i + f_{j,k,l-1}^i}{(\Delta r)^2} \\ \frac{\partial^2 f}{\partial S \partial V} &\simeq \frac{f_{j+1,k+1,l}^i - f_{j+1,k-1,l}^i - f_{j-1,k+1,l}^i + f_{j-1,k-1,l}^i}{4\Delta V \Delta S} \end{aligned}$$

Plugging these approximations into our PDE we arrive at:

$$\begin{aligned}
r_l f_{j,k,l}^{i+\frac{1}{3}} &= \frac{1}{2} V_k S_j^2 \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j-1,k,l}^{i+\frac{1}{3}}}{(\Delta S)^2} \right] + r_l S_j \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \right] \\
&+ \rho \sigma_v V_k S_j \left[\frac{f_{j+1,k+1,l}^i - f_{j+1,k-1,l}^i - f_{j-1,k+1,l}^i + f_{j-1,k-1,l}^i}{4\Delta V \Delta S} \right] \\
&+ \frac{1}{2} \sigma_v^2 V_k \left[\frac{f_{j,k+1,l}^i - 2f_{j,k,l}^i + f_{j,k-1,l}^i}{(\Delta V)^2} \right] + [\theta_v - \kappa_{v3} V_k] \left[\frac{f_{j,k+1,l}^i - f_{j,k-1,l}^i}{2\Delta V} \right] \\
&+ \frac{1}{2} \sigma_r^2 r_l \left[\frac{f_{j,k,l+1}^i - 2f_{j,k,l}^i + f_{j,k,l-1}^i}{(\Delta r)^2} \right] + [\theta_r - \kappa_{r3} r_l] \left[\frac{f_{j,k,l+1}^i - f_{j,k,l-1}^i}{2\Delta r} \right] \\
&- \left[\frac{f_{j,k,l}^{i+\frac{1}{3}} - f_{j,k,l}^i}{(\Delta \tau / 3)} \right]
\end{aligned}$$

Now rearrange this equation such that all terms involving τ step ($i + \frac{1}{3}$) are on the left hand side:

$$\begin{aligned}
&f_{j+1,k,l}^{i+\frac{1}{3}} \left[\frac{-V_k S_j^2}{2(\Delta S)^2} - \frac{r_l S_j}{2\Delta S} \right] + f_{j,k,l}^{i+\frac{1}{3}} \left[r_l + \frac{V_k S_j^2}{(\Delta S)^2} + \frac{1}{(\Delta \tau / 3)} \right] \\
&+ f_{j-1,k,l}^{i+\frac{1}{3}} \left[\frac{r_l S_j}{2\Delta S} - \frac{V_k S_j^2}{2(\Delta S)^2} \right] \\
&= f_{j,k,l}^i \left[\frac{-\sigma_v^2 V_k}{(\Delta V)^2} - \frac{\sigma_r^2 r_l}{(\Delta r)^2} + \frac{1}{(\Delta \tau / 3)} \right] + f_{j,k+1,l}^i \left[\frac{\sigma_v^2 V_k}{2(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} \right] \\
&+ f_{j,k-1,l}^i \left[\frac{\sigma_v^2 V_k}{2(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} \right] + f_{j,k,l+1}^i \left[\frac{\sigma_r^2 r_l}{2(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3} r_l)}{2\Delta r} \right] \\
&+ f_{j,k,l-1}^i \left[\frac{\sigma_r^2 r_l}{2(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3} r_l)}{2\Delta r} \right] \\
&+ \frac{\rho \sigma_v V_k S_j}{4\Delta V \Delta S} [f_{j+1,k+1,l}^i - f_{j+1,k-1,l}^i - f_{j-1,k+1,l}^i + f_{j-1,k-1,l}^i]
\end{aligned} \tag{6.1}$$

We see that everything on the right hand side of the above equation is known, whereas those terms on the left hand side, being those that involve time step ($i + \frac{1}{3}$), are unknown. However, notice that only the stock price dimension is varying on the left hand side of the equation; in other words, all of the time, volatility and interest

rate dimensions are fixed. This is typical of the ADI method and, as discussed before, allows us to reduce this potentially multi-dimensional problem into one dimension. Using the ADI method in this scenario will lead us to solve a $J \times J$ tri-diagonal matrix for each level of volatility and interest rates within the finite difference grid. The ADI method is computationally efficient as we only have to solve $KL \cdot J \times J$ systems of equations as opposed to one $JKL \times JKL$ equation system.

6.2 The second third of the step.

Once the values for the first auxillary time step have been calculated, we then need to complete the second third of the step. This involves moving to time step $(i + \frac{2}{3})$ from time step $(i + \frac{1}{3})$. The finite difference approximations at this point will be:

$$\begin{aligned} \frac{\partial f}{\partial \tau} &\simeq \frac{f_{j,k,l}^{i+\frac{2}{3}} - f_{j,k,l}^{i+\frac{1}{3}}}{(\Delta\tau/3)} \\ \frac{\partial f}{\partial S} &\simeq \frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \\ \frac{\partial^2 f}{\partial S^2} &\simeq \frac{f_{j+1,k,l}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j-1,k,l}^{i+\frac{1}{3}}}{(\Delta S)^2} \\ \frac{\partial f}{\partial V} &\simeq \frac{f_{j,k+1,l}^{i+\frac{2}{3}} - f_{j,k-1,l}^{i+\frac{2}{3}}}{2\Delta V} \\ \frac{\partial^2 f}{\partial V^2} &\simeq \frac{f_{j,k+1,l}^{i+\frac{2}{3}} - 2f_{j,k,l}^{i+\frac{2}{3}} + f_{j,k-1,l}^{i+\frac{2}{3}}}{(\Delta V)^2} \\ \frac{\partial f}{\partial r} &\simeq \frac{f_{j,k,l+1}^{i+\frac{1}{3}} - f_{j,k,l-1}^{i+\frac{1}{3}}}{2\Delta r} \\ \frac{\partial^2 f}{\partial r^2} &\simeq \frac{f_{j,k,l+1}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j,k,l-1}^{i+\frac{1}{3}}}{(\Delta r)^2} \\ \frac{\partial^2 f}{\partial S \partial V} &\simeq \frac{f_{j+1,k+1,l}^{i+\frac{1}{3}} - f_{j+1,k-1,l}^{i+\frac{1}{3}} - f_{j-1,k+1,l}^{i+\frac{1}{3}} + f_{j-1,k-1,l}^{i+\frac{1}{3}}}{4\Delta V \Delta S} \end{aligned}$$

Recalling that the PDE is:

$$\begin{aligned}
rf = & \frac{1}{2}VS^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} + \rho\sigma_vVS\frac{\partial^2 f}{\partial S\partial V} + \frac{1}{2}\sigma_v^2V\frac{\partial^2 f}{\partial V^2} + [\theta_v - \kappa_{v3}V]\frac{\partial f}{\partial V} \\
& + \frac{1}{2}\sigma_r^2r\frac{\partial^2 f}{\partial r^2} + [\theta_r - \kappa_{r3}r]\frac{\partial f}{\partial r} - \frac{\partial f}{\partial \tau}
\end{aligned}$$

We can then substitute these finite difference approximations into the PDE to obtain:

$$\begin{aligned}
r_l f_{j,k,l}^{i+\frac{1}{3}} = & \frac{1}{2}V_k S_j^2 \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j-1,k,l}^{i+\frac{1}{3}}}{(\Delta S)^2} \right] + r_l S_j \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \right] \\
& + \rho\sigma_v V_k S_j \left[\frac{f_{j+1,k+1,l}^{i+\frac{1}{3}} - f_{j+1,k-1,l}^{i+\frac{1}{3}} - f_{j-1,k+1,l}^{i+\frac{1}{3}} + f_{j-1,k-1,l}^{i+\frac{1}{3}}}{4\Delta V \Delta S} \right] \\
& + \frac{1}{2}\sigma_v^2 V_k \left[\frac{f_{j,k+1,l}^{i+\frac{2}{3}} - 2f_{j,k,l}^{i+\frac{2}{3}} + f_{j,k-1,l}^{i+\frac{2}{3}}}{(\Delta V)^2} \right] + [\theta_v - \kappa_{v3}V_k] \left[\frac{f_{j,k+1,l}^{i+\frac{2}{3}} - f_{j,k-1,l}^{i+\frac{2}{3}}}{2\Delta V} \right] \\
& + \frac{1}{2}\sigma_r^2 r_l \left[\frac{f_{j,k,l+1}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j,k,l-1}^{i+\frac{1}{3}}}{(\Delta r)^2} \right] + [\theta_r - \kappa_{r3}r_l] \left[\frac{f_{j,k,l+1}^{i+\frac{1}{3}} - f_{j,k,l-1}^{i+\frac{1}{3}}}{2\Delta r} \right] \\
& - \left[\frac{f_{j,k,l}^{i+\frac{2}{3}} - f_{j,k,l}^{i+\frac{1}{3}}}{(\Delta\tau/3)} \right]
\end{aligned}$$

Now rearrange this equation such that all terms involving time step $(i + \frac{2}{3})$ are on

the left hand side of the equation:

$$\begin{aligned}
& f_{j,k+1,l}^{i+\frac{2}{3}} \left[\frac{-\sigma_v^2 V_k}{2(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} \right] + f_{j,k,l}^{i+\frac{2}{3}} \left[\frac{1}{(\Delta\tau/3)} + \frac{\sigma_v^2 V_k}{(\Delta V)^2} \right] \\
& + f_{j,k-1,l}^{i+\frac{2}{3}} \left[\frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} - \frac{\sigma_v^2 V_k}{2(\Delta V)^2} \right] \\
& = f_{j,k,l}^{i+\frac{1}{3}} \left[-r_l - \frac{V_k S_j^2}{(\Delta S)^2} - \frac{\sigma_r^2 r_l}{(\Delta r)^2} + \frac{1}{(\Delta\tau/3)} \right] + f_{j+1,k,l}^{i+\frac{1}{3}} \left[\frac{r_l S_j}{2\Delta S} + \frac{V_k S_j^2}{2(\Delta S)^2} \right] \\
& + f_{j-1,k,l}^{i+\frac{1}{3}} \left[\frac{V_k S_j^2}{2(\Delta S)^2} - \frac{r_l S_j}{2\Delta S} \right] + f_{j,k,l+1}^{i+\frac{1}{3}} \left[\frac{\sigma_r^2 r_l}{2(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3} r_l)}{2\Delta r} \right] \\
& + f_{j,k,l-1}^{i+\frac{1}{3}} \left[\frac{\sigma_r^2 r_l}{2(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3} r_l)}{2\Delta r} \right] \\
& + \frac{\rho\sigma_v V_k S_j}{4\Delta V \Delta S} \left[f_{j+1,k+1,l}^{i+\frac{1}{3}} - f_{j+1,k-1,l}^{i+\frac{1}{3}} - f_{j-1,k+1,l}^{i+\frac{1}{3}} + f_{j-1,k-1,l}^{i+\frac{1}{3}} \right]
\end{aligned} \tag{6.2}$$

Again, observe that all terms to the right of the equals sign are known, whereas all those terms on the left, being those that involve time step $(i + \frac{2}{3})$, are unknown. Additionally, of those terms which are unknown, only the volatility dimension is varying.

6.3 The final third of the step.

The finite difference approximations will be much the same as before for this final step, except that the approximations for stock price and volatility will be done at time step $(i + \frac{2}{3})$, whilst the approximations for interest rates will be done at time

step $(i + 1)$. This results in the following set of FDAs:

$$\begin{aligned}
\frac{\partial f}{\partial \tau} &\simeq \frac{f_{j,k,l}^{i+1} - f_{j,k,l}^{i+\frac{2}{3}}}{(\Delta\tau/3)} \\
\frac{\partial f}{\partial S} &\simeq \frac{f_{j+1,k,l}^{i+\frac{2}{3}} - f_{j-1,k,l}^{i+\frac{2}{3}}}{2\Delta S} \\
\frac{\partial^2 f}{\partial S^2} &\simeq \frac{f_{j+1,k,l}^{i+\frac{2}{3}} - 2f_{j,k,l}^{i+\frac{2}{3}} + f_{j-1,k,l}^{i+\frac{2}{3}}}{(\Delta S)^2} \\
\frac{\partial f}{\partial V} &\simeq \frac{f_{j,k+1,l}^{i+\frac{2}{3}} - f_{j,k-1,l}^{i+\frac{2}{3}}}{2\Delta V} \\
\frac{\partial^2 f}{\partial V^2} &\simeq \frac{f_{j,k+1,l}^{i+\frac{2}{3}} - 2f_{j,k,l}^{i+\frac{2}{3}} + f_{j,k-1,l}^{i+\frac{2}{3}}}{(\Delta V)^2} \\
\frac{\partial f}{\partial r} &\simeq \frac{f_{j,k,l+1}^{i+1} - f_{j,k,l-1}^{i+1}}{2\Delta r} \\
\frac{\partial^2 f}{\partial r^2} &\simeq \frac{f_{j,k,l+1}^{i+1} - 2f_{j,k,l}^{i+1} + f_{j,k,l-1}^{i+1}}{(\Delta r)^2} \\
\frac{\partial^2 f}{\partial S \partial V} &\simeq \frac{f_{j+1,k+1,l}^{i+\frac{2}{3}} - f_{j+1,k-1,l}^{i+\frac{2}{3}} - f_{j-1,k+1,l}^{i+\frac{2}{3}} + f_{j-1,k-1,l}^{i+\frac{2}{3}}}{4\Delta V \Delta S}
\end{aligned}$$

Substituting into the PDE we get:

$$\begin{aligned}
r_l f_{j,k,l}^{i+\frac{2}{3}} &= \frac{1}{2} V_k S_j^2 \left[\frac{f_{j+1,k,l}^{i+\frac{2}{3}} - 2f_{j,k,l}^{i+\frac{2}{3}} + f_{j-1,k,l}^{i+\frac{2}{3}}}{(\Delta S)^2} \right] + r_l S_j \left[\frac{f_{j+1,k,l}^{i+\frac{2}{3}} - f_{j-1,k,l}^{i+\frac{2}{3}}}{2\Delta S} \right] \\
&+ \rho \sigma_v V_k S_j \left[\frac{f_{j+1,k+1,l}^{i+\frac{2}{3}} - f_{j+1,k-1,l}^{i+\frac{2}{3}} - f_{j-1,k+1,l}^{i+\frac{2}{3}} + f_{j-1,k-1,l}^{i+\frac{2}{3}}}{4\Delta V \Delta S} \right] \\
&+ \frac{1}{2} \sigma_v^2 V_k \left[\frac{f_{j,k+1,l}^{i+\frac{2}{3}} - 2f_{j,k,l}^{i+\frac{2}{3}} + f_{j,k-1,l}^{i+\frac{2}{3}}}{(\Delta V)^2} \right] + [\theta_v - \kappa_{v3} V_k] \left[\frac{f_{j,k+1,l}^{i+\frac{2}{3}} - f_{j,k-1,l}^{i+\frac{2}{3}}}{2\Delta V} \right] \\
&+ \frac{1}{2} \sigma_r^2 r_l \left[\frac{f_{j,k,l+1}^{i+1} - 2f_{j,k,l}^{i+1} + f_{j,k,l-1}^{i+1}}{(\Delta r)^2} \right] + [\theta_r - \kappa_{r3} r_l] \left[\frac{f_{j,k,l+1}^{i+1} - f_{j,k,l-1}^{i+1}}{2\Delta r} \right] \\
&- \left[\frac{f_{j,k,l}^{i+1} - f_{j,k,l}^{i+\frac{2}{3}}}{(\Delta\tau/3)} \right]
\end{aligned}$$

Finally, rearrange this equation such that all terms involving time step $(i + 1)$ are on the left hand side:

$$\begin{aligned}
& f_{j,k,l+1}^{i+1} \left[\frac{-\sigma_r^2 r_l}{2(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3} r_l)}{2\Delta r} \right] + f_{j,k,l}^{i+1} \left[\frac{\sigma_r^2 r_l}{(\Delta r)^2} + \frac{1}{(\Delta \tau/3)} \right] \\
& + f_{j,k,l-1}^{i+1} \left[\frac{-\sigma_r^2 r_l}{2(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3} r_l)}{2(\Delta r)} \right] \\
& = f_{j,k,l}^{i+\frac{2}{3}} \left[-r_l - \frac{V_k S_j^2}{(\Delta S)^2} - \frac{\sigma_v^2 V_k}{(\Delta V)^2} + \frac{1}{(\Delta \tau/3)} \right] + f_{j+1,k,l}^{i+\frac{2}{3}} \left[\frac{V_k S_j^2}{2(\Delta S)^2} + \frac{r_l S_j}{2\Delta S} \right] \\
& + f_{j-1,k,l}^{i+\frac{2}{3}} \left[\frac{V_k S_j^2}{2(\Delta S)^2} - \frac{r_l S_j}{2\Delta S} \right] + f_{j,k+1,l}^{i+\frac{2}{3}} \left[\frac{\sigma_v^2 V_k}{2(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} \right] \\
& + f_{j,k-1,l}^{i+\frac{2}{3}} \left[\frac{-(\theta_v - \kappa_{v3} V_k)}{2\Delta V} + \frac{\sigma_v^2 V_k}{2(\Delta V)^2} \right] \\
& + \frac{\rho \sigma_v V_k S_j}{4\Delta V \Delta S} \left[f_{j+1,k+1,l}^{i+\frac{2}{3}} - f_{j+1,k-1,l}^{i+\frac{2}{3}} - f_{j-1,k+1,l}^{i+\frac{2}{3}} + f_{j-1,k-1,l}^{i+\frac{2}{3}} \right]
\end{aligned} \tag{6.3}$$

6.4 Boundary conditions.

If we are not on one of the boundaries for our finite difference grid, then the equations relating to the first, second and third parts of the step are given by (6.1), (6.2) and (6.3). For ease of notation, we shall write the above equations more concisely as:

The first part of the step:

$$\begin{aligned}
& f_{j+1,k,l}^{i+\frac{1}{3}} \xi_1 + f_{j,k,l}^{i+\frac{1}{3}} \xi_2 + f_{j-1,k,l}^{i+\frac{1}{3}} \xi_3 = f_{j,k,l}^i \xi_4 + f_{j,k+1,l}^i \xi_5 + f_{j,k-1,l}^i \xi_6 + f_{j,k,l+1}^i \xi_7 \\
& + f_{j,k,l-1}^i \xi_8 + \left[f_{j+1,k+1,l}^i - f_{j+1,k-1,l}^i - f_{j-1,k+1,l}^i + f_{j-1,k-1,l}^i \right] \xi_9
\end{aligned}$$

The second third of a step:

$$\begin{aligned}
& f_{j,k+1,l}^{i+\frac{2}{3}} \xi_{10} + f_{j,k,l}^{i+\frac{2}{3}} \xi_{11} + f_{j,k-1,l}^{i+\frac{2}{3}} \xi_{12} = f_{j,k,l}^{i+\frac{1}{3}} \xi_{13} + f_{j+1,k,l}^{i+\frac{1}{3}} \xi_{14} + f_{j-1,k,l}^{i+\frac{1}{3}} \xi_{15} + f_{j,k,l+1}^{i+\frac{1}{3}} \xi_{16} \\
& + f_{j,k,l-1}^{i+\frac{1}{3}} \xi_{17} + \left[f_{j+1,k+1,l}^{i+\frac{1}{3}} - f_{j+1,k-1,l}^{i+\frac{1}{3}} - f_{j-1,k+1,l}^{i+\frac{1}{3}} + f_{j-1,k-1,l}^{i+\frac{1}{3}} \right] \xi_{18}
\end{aligned}$$

The final third of a step:

$$\begin{aligned}
 f_{j,k,l+1}^{i+1}\xi_{19} + f_{j,k,l}^{i+1}\xi_{20} + f_{j,k,l-1}^{i+1}\xi_{21} &= f_{j,k,l}^{i+\frac{2}{3}}\xi_{22} + f_{j+1,k,l}^{i+\frac{2}{3}}\xi_{23} + f_{j-1,k,l}^{i+\frac{2}{3}}\xi_{24} + f_{j,k+1,l}^{i+\frac{2}{3}}\xi_{25} \\
 &+ f_{j,k-1,l}^{i+\frac{2}{3}}\xi_{26} + \left[f_{j+1,k+1,l}^{i+\frac{2}{3}} - f_{j+1,k-1,l}^{i+\frac{2}{3}} - f_{j-1,k+1,l}^{i+\frac{2}{3}} + f_{j-1,k-1,l}^{i+\frac{2}{3}} \right] \xi_{27}
 \end{aligned}$$

Note that we need to consider what happens at the boundaries of our finite difference mesh. Obviously, we can't refer to points outside the grid, as such points don't exist, so we need to redefine our finite difference approximations when this is an issue. The time to maturity dimension (τ) won't pose a problem. However, there are still the stock price (j), volatility (k) and interest rate (l) dimensions to consider. The possibilities, put simply, will be as follows:

Stock Price:

- Upper boundary (S_J)
- Not on a boundary.
- Lower boundary (S_0)

Volatility:

- Upper boundary (V_K)
- Not on a boundary.
- Lower boundary (V_0)

Interest rates:

- Upper boundary (r_L)

- Not on a boundary.
- Lower boundary (r_0)

See the discussion in the Stochastic Volatility (SV) case as to how we deal with the FDAs for stock price and volatility at the boundaries. However, we still have the task, at least in this case, of considering what happens at the boundaries for interest rates. If we are not on an interest rate boundary within the finite difference mesh, then adjusting our FDAs relating to interest rates is a non-issue. However, consider what happens when interest rates become either very large or very small. As interest rates become very large, the discount rate applied to future payoffs will likewise be getting larger, and the option's value, $f(S, V, r, T)$, will tend towards zero as interest rates move towards infinity (ie. $f(S, V, \infty, T) = 0$). Any additional increments or decrements in interest rates will have little impact on the option's value at this point. In other words, we could reasonably expect $\frac{\partial f}{\partial r}$ to be relatively constant for very high levels of the interest rate, suggesting that $\frac{\partial^2 f}{\partial r^2} \simeq 0$. As for very low levels of the interest rate, as r tends towards zero the term $\frac{1}{2}\sigma_r^2 r \frac{\partial^2 f}{\partial r^2}$ will become smaller and relatively insignificant, assuming that $\frac{\partial^2 f}{\partial r^2}$ does not increase drastically. In fact, it would be reasonable to suppose that $\frac{\partial^2 f}{\partial r^2} = 0$ at the lowest point for r in the finite difference mesh. As for what happens to $\frac{\partial f}{\partial r}$ at the lower boundary for r , we shall have to give it a new definition such that we don't refer to points outside of the finite difference mesh.

For a more complete instruction on how to implement this numerical procedure at the finite difference mesh boundaries, please refer to appendix C.

Chapter 7

An example of stochastic volatility and stochastic interest rates in options pricing.

We are now going to run through a hypothetical example of stock options pricing using the ADI method with the parameters in table 7.1. Figure 7.1 and 7.2 represent the derivative prices and hedging sensitivities for different levels of the stock price, volatility and interest rates.

Notice in figure 7.1(A) that the call option's value is positively related to the stock price, with the opposite being true for the put option. Given that a European call option's payoff at maturity is $\max(S_T - K, 0)$, whilst the payoff for an otherwise equivalent put option is $\max(K - S_T, 0)$, it is hardly surprising that the graph for call options slopes upward in the stock price dimension whilst the opposite is true for put options. In short, a higher stock price, *ceteris paribus*, is favourable for call option holders and unfavourable for put options holders. However, notice that both graphs have only a very slight slope for heavily OTM options. It is unlikely that such options will swing into the money, given volatility, meaning that a change in the value of the underlying asset has only a very minor effect on the value of the option

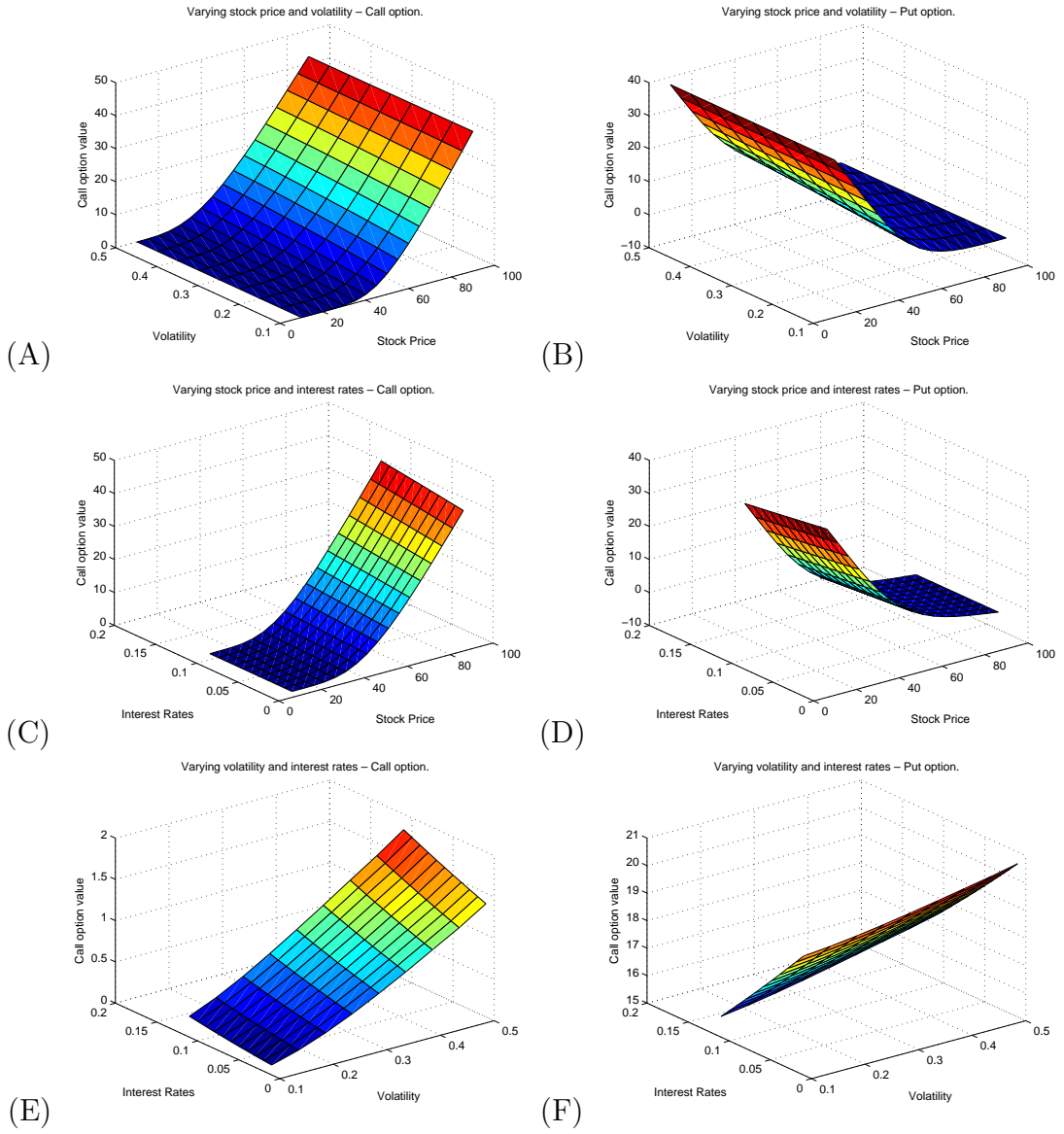


Figure 7.1: One dimension, being either stock price (S), volatility (V), or interest rates (r) is fixed for each of the graphs. Interest rate=.05, volatility=.3, and stock price=50 for each of rows one, two and three respectively.

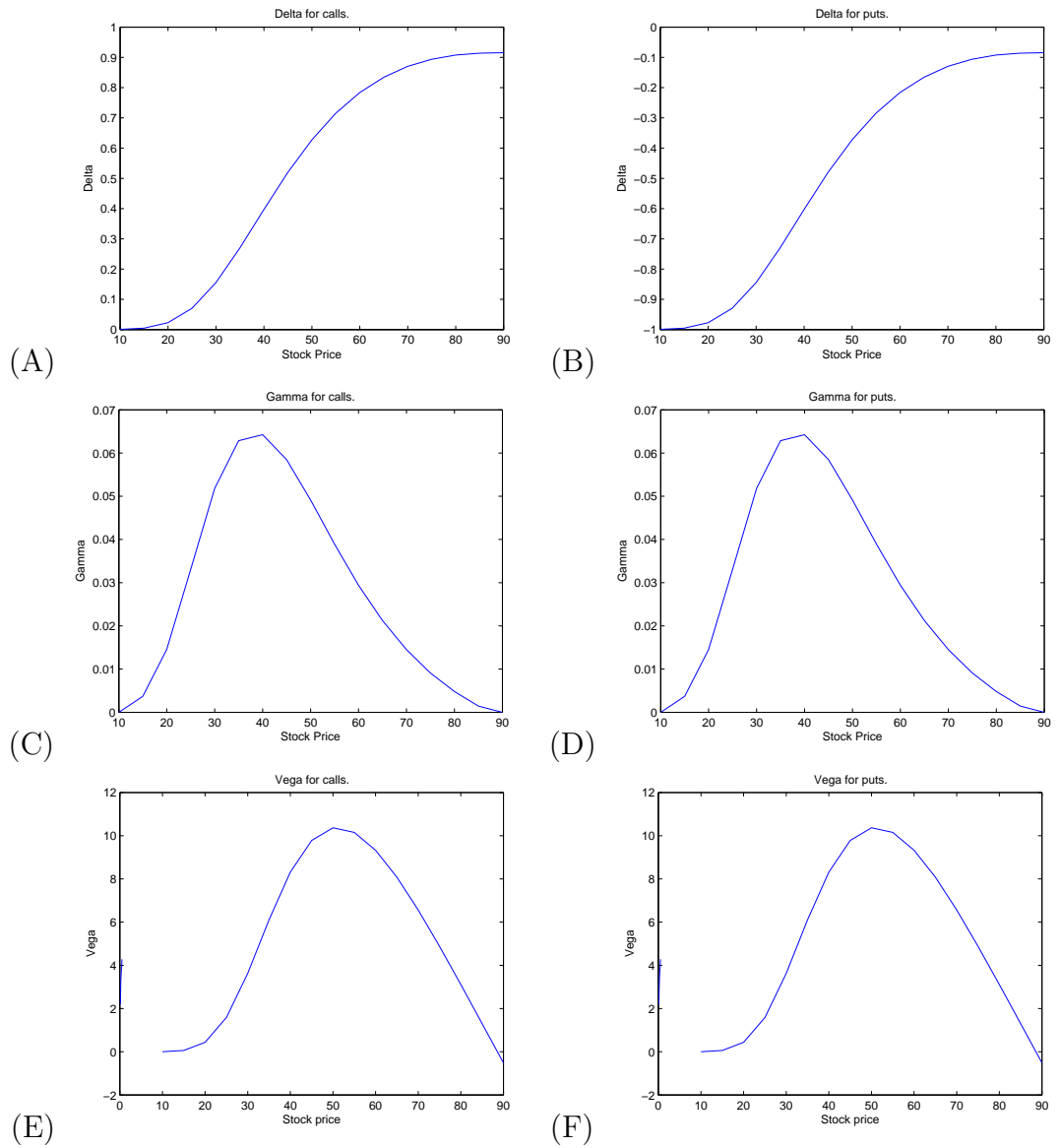


Figure 7.2: Volatility and interest rates are set equal to .3 and .05 respectively for each of these hedging sensitivities graphs.

Table 7.1: These hypothetical parameter values are used to produce the graphs that are to follow.

Parameter values.	
$\Delta t = .01$	$\rho_{sv} = -.3$
$\Delta s = 5$	$K = 50$
$\Delta v = .05$	$\sigma_r = .05$
$\Delta r = .01$	$\kappa_r = .5$
$T = 1$	$\theta_r = .02$
$Smin = 10$	$\sigma_v = .2$
$Smax = 90$	$\kappa_v = 2$
$Vmin = .10$	$\theta_v = .08$
$Vmax = .50$	$Rmax = .11$
$Rmin = .01$	

at these points. Indeed, this is reflected in our calculations for delta ($\Delta = \frac{\partial f}{\partial S}$) in figure 7.2(A) and (B). As an option moves more OTM, the delta of such an option will move closer towards zero and the option's value will be less sensitive to changes in the underlying asset. The shape of the graph for delta looks much the same for call and put options, the difference being that Δ values range between 0 and 1 for call options, whilst the range for put options is between 0 and -1 . Again, this is no surprise. Delta values for calls ($\frac{\partial c}{\partial S}$) are positive because a higher underlying asset price benefits call option holders, and the delta of a put option ($\frac{\partial p}{\partial S}$) is negative because a higher underlying asset price will result in a decline in put option value.

Hence, as the underlying asset price changes so will the corresponding delta of the option, and the rate of change for delta is known as gamma $\Gamma = (\frac{\partial^2 f}{\partial S^2})$. The underlying asset will always have a Δ of 1, and consequently a Γ of zero. Because the delta of options will change over time, periodic rebalancing of our option positions would be required should we wish to maintain a delta neutral portfolio. These sensitivities can also be used to create synthetic options through replication. However, this does raise some concerns. Consider the case of a portfolio manager wishing to create a protective put position synthetically over an indexed equity portfolio. This

type of ‘portfolio insurance’ is a means of protecting the portfolio’s value should an adverse fall in equity values occur. Further suppose that the replicated option was approximately ATM, meaning that the gamma of the option being replicated is likely to be large. Would the synthetic put option be effective in the case of a market crash, for example? The answer, of course, is no. Replicating the delta of an option is only effective in the case of small movements in the underlying asset. Delta tells only part of the story. Gamma attempts to quantify the rate of change in delta, and therefore gives an indication of how often we might expect to rebalance a portfolio in order to maintain approximate delta neutrality. We can see from figure 7.2(C) and (D) that Γ is identical for both European put and call options with the same underlying parameters. Most of the change in Δ occurs when the option is approximately ATM, and then delta becomes relatively constant ($\Gamma \simeq 0$) for options that are either heavily ITM or heavily OTM. There are two reasons for this; firstly, heavily OTM options are likely to expire worthless anyway, and slight changes in the underlying asset aren’t going to make much difference to the option’s value ($\Delta \simeq 0$ and $\Gamma \simeq 0$); secondly, options that are heavily ITM are most likely to be exercised, meaning that the value of a heavily ITM European call option today, for example, might be approximated by $(S_t - Ke^{-rT})$, such that $\Delta \simeq 1$ and $\Gamma \simeq 0$. Finally, notice that Γ is always positive, meaning that delta is always increasing as the underlying asset price increases.

Looking at the graphs in 7.1(A) and (B) we can see that the value of an option is positively related to volatility; in other words, higher levels of volatility result in a higher option value. Keep in mind that volatility in our SVSI model follows a mean reverting process $\left[dV = [\theta_v - \kappa_v V] dt + \sigma_v \sqrt{V} dz_2 \right]$. Regardless of the current level of volatility, reversion will always occur towards its long term theoretical level in this model. Consequently, changes in volatility, whilst still having a positive effect on option value, will not have such a pronounced effect when compared to a permanent volatility increase. Also notice that the level of vega ($= \frac{\partial f}{\partial V}$) in figure 7.2(E) and (F) is identical and always positive for European puts and calls. A higher level of volatility gives all options a better chance of swinging into the money. Investors taking a long position in puts and calls are less concerned about the downside of volatility since they cannot lose more than the option premium. However, heavily

ITM options still do have a significant downside potential, meaning that additional volatility increments for these options are unlikely to add significantly to value. Likewise, options sitting deeply OTM are unlikely to be exercised, regardless of whether volatility becomes very large, and hence vega is relatively small and insignificant. These facts are evidenced by vega dropping off at the extremities for stock price. Note that the underlying asset always has a vega of zero, as movements in the underlying are fully explained by its delta.

Lastly, let's turn our attention to the effect of interest rates on the value of puts and calls. Figure 7.1(E) and (F) shows that, at least in this case, volatility seems far more important in determining the value of these basic derivatives. This can be seen by comparing the slope of the steeper volatility dimension to that of the interest rate dimension. However, we are dealing with only one year as our time to maturity. Indeed, most stocks expire in nine months or less. Consequently, we wouldn't expect discounting to play such a large role for these comparatively short maturity instruments. When we turn our attention to convertible bonds, however, which tend to have maturity dates spanning many years into the future, then interest rates are likely to play a much more important role. Higher interest rates will result in a higher drift term for the stock price in the risk neutral world, and this will result in higher values for call options and lower values for put options. As a result, the values for $\rho = \frac{\partial f}{\partial r}$ are positive for call options and negative for put options. The effect of discounting, on the other hand, will be to lower the value of both call and put options. ρ tends to be largest for deep ITM calls and OTM puts, and tends to be most sensitive to changes in the underlying asset price for ATM options. Again, the process followed by interest rates is mean reverting, so the overall effect of a change in interest rates is not as large as a permanent change.

Chapter 8

An example of stochastic volatility and stochastic interest rates in fixed income derivatives pricing.

We now do a similar exercise as the previous section, except that we are now going to use the ADI method to price fixed income instruments. The parameter values for the interest rate and firm value processes will be identical to those used in the previous section. Additionally, suppose that we are dealing with a firm that has issued one bond with a face value of \$40, convertible into ten percent of the outstanding stock post conversion ($z = .10$). Figure 8.1 summarises the results graphically.

Firstly, observe the relation between firm value and bond value in figure 8.1(D). Given that the aggregate face value of bonds outstanding is 40 ($l \cdot FV = 1 \times 40$), we can see that there is a ‘default region’ for firm values around and below this amount. Indeed, this section of the graph is very steep, as reflected by the high and rapidly declining value for delta in figure 8.1(E), and the value of each convertible bond in the event of default is equal to $(\frac{V}{7})$. Recall that the bonds, in aggregate, will be worth ten percent of the firm’s outstanding stock should conversion occur (i.e. $z = .10$). For very high levels of firm value, conversion is likely to

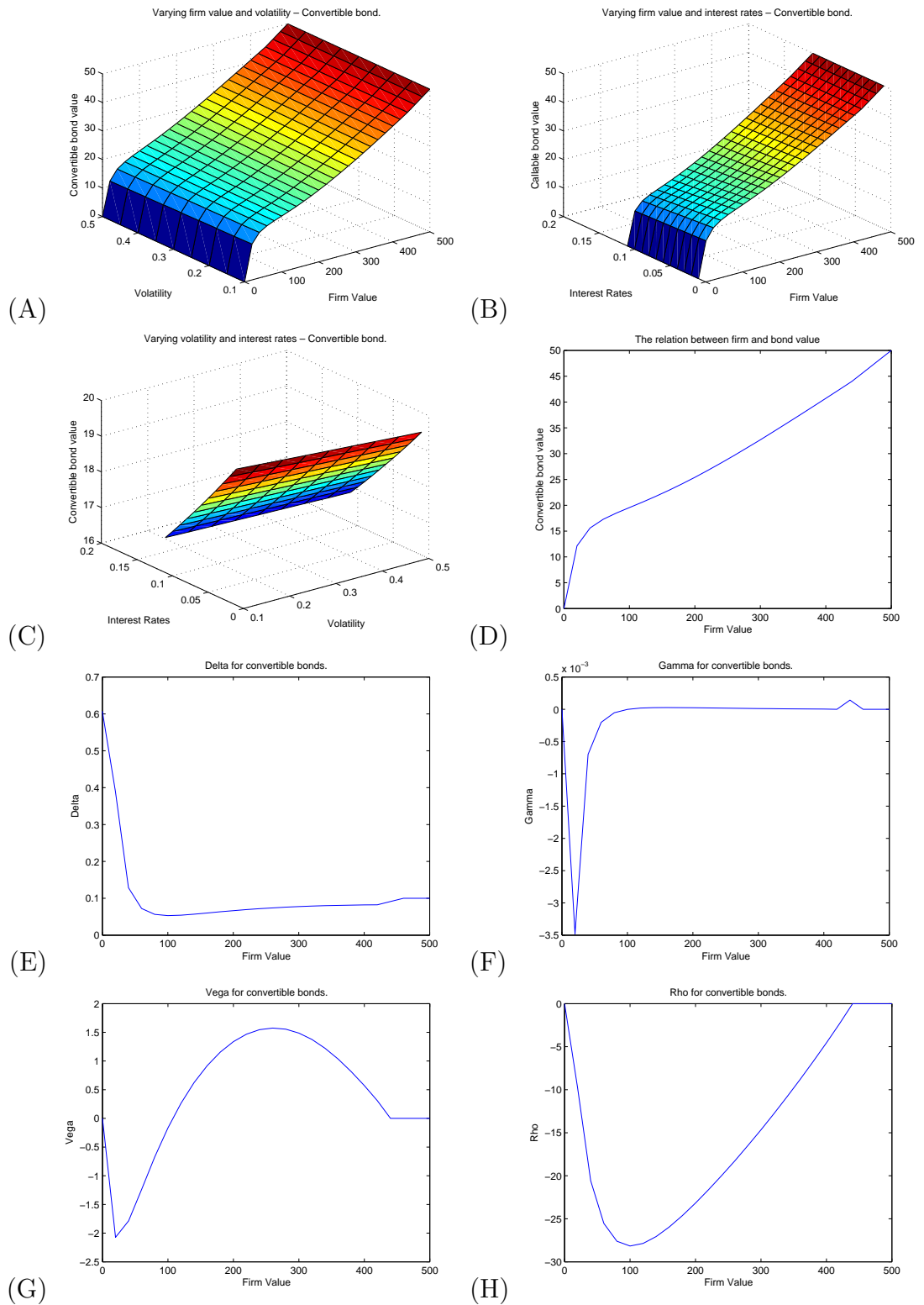


Figure 8.1: Here are some graphs relating to price and sensitivities for convertible bonds.

occur immediately, the bondholders then receiving ten percent of the outstanding stock post conversion, and the graph for delta becomes linear at this point with a slope of $f_v = z = .1$. We shall refer to this area as the ‘conversion region’. For levels of firm value above the default region and below the conversion region, we see that the value of each bond today is something less than its face value. Since we have not incorporated dividends or coupons in this example, there is only one future payoff, being the FV, and the bond’s value today is the present value of this amount.

We have already referred to figure 8.1(E) in discussing how the convertible bond’s value varies with firm value. With this understanding of Δ , an interpretation of Γ in figure 8.1(C)(2) follows quite naturally. In the default region where bond value is increasing sharply in relation to firm value, it is plain to see that Δ is large and positive. However, as we move out of the default region Δ becomes smaller and more steady. This suggests that Γ , the rate of change in Δ , is first negative before becoming relatively stable. As for the conversion region, this part of figure 8.1(D) is linear with a slope of $z = .1$, as reflected by a $\Delta = .1$ and a $\Gamma = 0$.

As to whether increased volatility adds value to the convertible bond holder, this will depend on the current level of firm value (i.e where we are on the curve in figure 8.1(D)). Given that the curve is concave over the default region, increased volatility whilst in this region will result in a decrease in convertible bond value. To see why this is the case, consider that a one unit movement downward in firm value whilst in the default region will result in a large decline in bond value, whereas a one unit increase in firm value will result in a relatively smaller increase in bond value. In short, increased volatility is not beneficial to the bond holder whilst in the default region. As firm value begins to increase (approximately ≥ 100 in this case), the curve in figure 8.1(D) begins to become convex and increased volatility is then beneficial from the bond holder’s perspective. Indeed, it appears that vega is greatest for firm values of around 260 in this example, as shown in figure 8.1(G). For really high levels of firm value where conversion occurs immediately, volatility has no bearing on this immediate payoff, and hence vega is zero.

Finally, let's turn our attention to ρ in figure 8.1(H). Because of the long maturity of fixed income instruments when compared to stock options, for example, we can logically expect that changes in interest rates are likely to have more of a pronounced effect on bonds. We can see that ρ is largely negative, meaning that higher interest rates are going to lower the bond's value, which comes as no surprise. A firm value of zero will result in bankruptcy and the bondholders recovering none of their investment; the bond's value is zero in this scenario, regardless of interest rates, and hence ρ is zero. As for the conversion region, we would expect that these bond's will be converted immediately, effectively removing the time value of money from consideration. As a result, ρ is also zero for very high levels of firm value.

Chapter 9

Data.

For the purposes of data collection, we have used several sources. Firstly, the Merger BondSource Corporate Bond Securities Database has provided a wealth of information regarding different bond issues, their trading prices and characteristics. We can then apply the numerical techniques described earlier to assess which approach is most effective in derivatives pricing. Other papers, namely Bakshi et al. (1997), have assessed the relative merits of different options pricing models in a stock options context. The findings of papers working with stock options, however, are not necessarily transferrable to convertible bonds, and therein lies the benefit of our study. The unique characteristics of convertible bonds, in particular the longer maturity, could reasonably be expected to have a significant influence on the appropriate pricing approach. We conjecture that, whilst adding a stochastic interest rate element may not be significantly beneficial in pricing short maturity derivatives, the longer maturity of convertible bonds necessitates a slightly more complex SVSI model. Secondly, through the St. Louis Fed FRED (Federal Reserve Economic Database) we were able to obtain historical information on interest rates. Given this data, we then undertook GMM procedures to back out parameters necessary for pricing $(\theta_v, \kappa_v, \sigma_v, \rho_{sv}, \theta_r, \kappa_r, \sigma_r)$.

For our interest rate data, we used daily treasury bill rates covering the period from

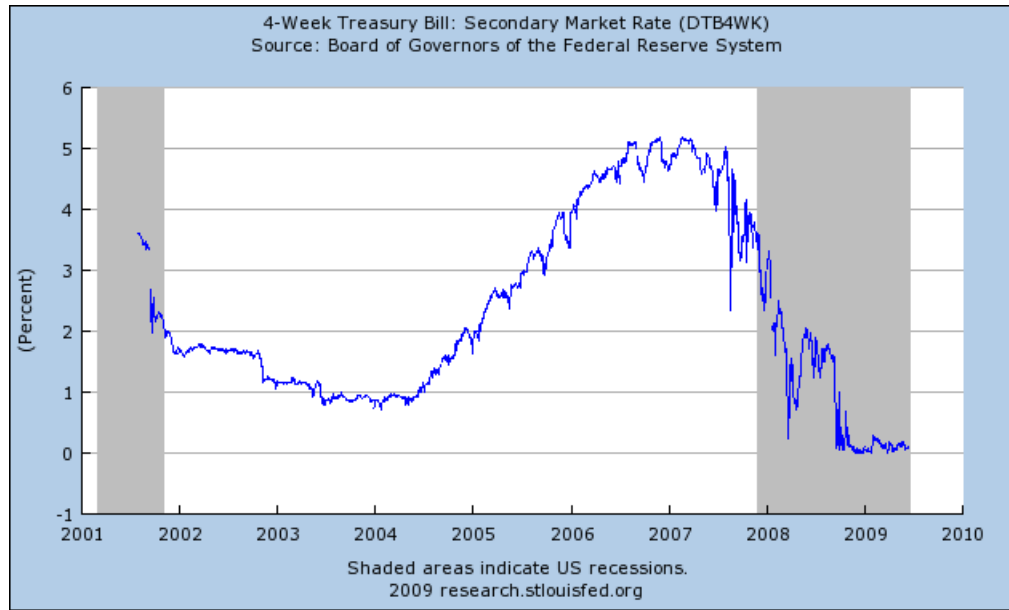


Figure 9.1: Treasury bill rates obtained from the St. Louis Federal Reserve Economic Database (FRED).

31/07/2001 to 27/03/2009. A graph of this data can be seen below. Recall that the process for interest rates has been specified as: $dr = [\theta_r - \kappa_r r] dt + \sigma_r \sqrt{r} dz_3$. We can then make the following definition $r_{t+1} = r_t + (\theta_r - \kappa_r r_t) \cdot \Delta t + \varepsilon_t$ such that:

$$\varepsilon_t = r_{t+1} - r_t - (\theta_r - \kappa_r r_t) \cdot \Delta t$$

We would then expect that:

- $E(\varepsilon_t) = 0$
- $E(\varepsilon_t \cdot r_t) = 0$
- $E(\varepsilon_t^2) = \sigma_r^2 r_t \cdot \Delta t$

We can then choose values for θ_r , κ_r and σ_r such that we minimise the squared

Table 9.1: Estimated parameters for the interest rate process.

Variable	value
θ_r	.0074
κ_r	.6054
σ_r	.0632

deviation of the moments from their theoretical values. In other words, we wish to minimise $[(E(\varepsilon_t) - 0)^2 + (E(\varepsilon \cdot r_t) - 0)^2 + (E(\varepsilon^2) - \sigma_r^2 \cdot r_t \Delta t)^2]$ by choosing appropriate parameter values. One way to do this is to use the ‘lsqnonlin’ command in Matlab, guess the initial parameter values to begin with, and then let the computer run its optimisation routine. The result is values for θ_r , κ_r and σ_r which can be used later in pricing convertible bonds. Based on the FRED data, the estimated parameter values relating to the interest rate process are given in table 9.1.

Estimating the parameters associated with firm value and volatility, however, is significantly more troublesome. First, we need a data series of the firm value in order to undertake our estimation procedures. Firm value is simply the sum of all equity and debt claims. Unfortunately, the debt claims tend to be traded rather infrequently; in other words, data relating to the marked value of debt claims is sparse. On the other hand, we have equity claims that are thickly traded and historical market price information that is easily accessible. Not only do we want a good frequency of firm value data, but we also want data that is reliable in order for the parameter estimates to be sensible. The approach we took was to assume that the value of debt claims was some constant, α , of the firm’s value. Letting S , B and E represent firm value, aggregate outstanding debt and total equity claims respectively, we can then express the relationship mathematically as $S = B + E$. Therefore:

$$\begin{aligned}
S &= \alpha \cdot S + E \\
S(1 - \alpha) &= E \\
S &= \frac{E}{(1 - \alpha)}
\end{aligned}$$

Given the stock price series, we can then divide each observation by $(1 - \alpha)$ to get a series for firm value. Of course, we need to estimate α first. Since $B = \alpha \cdot S$, then $\alpha = \frac{B}{S}$. In order to obtain an approximation of α , an average of total liabilities to total assets was taken from the financial statements; in short, we used book value approximations for B and S and then took an average of $\frac{B}{S}$ over a period of five years to estimate α . Given the approximate series for firm value, we then undertook a simulated GMM approach to estimate $\theta_v, \kappa_v, \sigma_v$, and ρ_{sv} . The approach to estimation can be explained step by step as follows:

1. Simulate two sets of T standard normal random variables, where T is a large number. (For this study we use $T = 1000000$).
 - $V_{t+1} = V_t + [\theta_v - \kappa_v V_t] \Delta t + \sigma_v \sqrt{V_t} \phi_2 \sqrt{\Delta t}$
 - $S_{t+1} = S_t + \mu S_t \Delta t + \sqrt{V_t} S_t \phi_1 \sqrt{\Delta t}$
2. Given the first set of normal variables and the equation for volatility, simulate a path for volatility. Recall that $\Delta z \sim N(0, \sqrt{\Delta t})$.
3. Given the simulated path for volatility and our second set of standard normal random variables, a simulated path for the firm value can be created.
4. Compare the actual moments from the data to the moments from our simulations. We can then choose values for $\theta_v, \kappa_v, \sigma_v$ and ρ_{sv} such that we minimise the sum of squared deviations of the moments from their simulated values. A computer can run through this optimisation routine by doing iterations from step two onwards.

The moments we are interested in are:

- $E\{[\Delta \ln(FV)]^2\}$
- $E\{[\Delta \ln(FV)]^3\}$
- $E\{[\Delta \ln(FV)]^4\}$
- $E\{[\Delta \ln(FV_t) \cdot \Delta \ln(FV_{t-1})]\}$

Keep in mind that parameters relating to the interest rate process will be estimated only once. However, the parameters relating to volatility and firm value will have to be estimated for each issuing firm. Given the parameters relating to firm value, volatility and interest rates accompanied with the information provided in the Merger BondSource Database, we are now in a position to calculate the theoretical values for convertible bonds issued by these firms. We concentrated on eight firms, and the parameters are displayed in table 9.2.

There appears to be significant variability in σ_v . Indeed, Chevron Corp. appears to have the lowest level of σ_v , suggesting that the level of volatility for this firm is unlikely to vary much through time. Consequently, one would expect that adding a stochastic volatility element to the pricing of Chevron's bonds is unlikely to add significantly to the accuracy of our pricing model in this extreme case. Interestingly, ρ is positive for all firms, contrary to what we might expect from the leverage effect. According to the leverage effect, higher levels of firm leverage add to the risk and volatility of the firm and therefore lower firm value, leading to the suggestion that ρ could be negative.

Table 9.2: Estimated parameter values for the firm value processes.

Parameter values.					
Firm:	Ticker Symbol	θ_v	κ_v	σ_v	ρ
McDonalds	MCD	.1764	1.6601	.3178	.333
AT&T Inc	T	.2081	1.3556	.1711	.4219
JPMorgan Chase	JPM	.2306	0.6779	.2772	.6610
Chevron Corp.	CVX	.1763	1.2006	.0003	.2244
General Electric	GE	.2194	1.2201	.1791	.4872
Cisco Systems	CSCO	.2336	0.6991	.1269	.7655
Intel	INTC	.2334	0.7432	.1082	.7431
Hewlett Packard	HPQ	.2175	0.7844	.0987	.5792

Chapter 10

Results.

We now take selected transactions from the database, pricing these bonds using several different models and then comparing how well each approach did in predicting the traded price. Firstly, note that we applied four categories of models:

- SVSI (Stochastic volatility and stochastic interest rates)
- SV (Stochastic volatility)
- SI (Stochastic interest rates)
- Brennan and Schwartz (Volatility and interest rates are assumed constant)

The data suggests that the SV/SVSI models are most accurate in general, whilst the simple Black-Scholes approach with assumed constant volatility and interest rates was the least accurate. The largest pricing error occurred using the Brennan and Schwartz model for the first transaction involving CVX. Although, with an average pricing error of approximately 7%, the Brennan and Schwartz model doesn't do such a bad job. Indeed, the SV/SVSI models didn't fare significantly better with an average pricing error of around 5.5%. Keeping this in mind, we can then explore the circumstances in which adding stochastic interest rates and volatility is definitely

Table 10.1: Traded prices of bonds accompanied by their estimated values according to different pricing models.

Prices.					
Firm ticker	Traded Price	SVSI	SV	SI	Brennan and Schwartz
MCD ₁	1000.00	987.35	1030.60	1008.70	1014.40
MCD ₁	987.45	1019.10	1059.10	1063.40	1060.70
MCD ₂	1000.00	1104.00	1073.80	1045.40	1062.20
MCD ₂	853.22	965.90	937.00	930.00	905.00
AT&T ₁	997.80	1003.30	923.16	1001.10	921.53
CVX ₁	1021.73	1064.60	1228.20	1064.90	1281.80
CVX ₁	1012.40	1026.10	1157.40	1025.90	1222.60
CVX ₁	1015.65	1014.00	1017.00	1010.70	1021.10
CVX ₁	1047.02	1034.70	1060.40	1029.30	1109.10
CSCO ₁	1038.25	973.52	974.71	993.10	995.70
CSCO ₁	1064.29	997.32	995.28	1001.20	997.44
CSCO ₁	1033.10	1012.10	1012.10	1012.10	1012.10
CSCO ₁	961.81	990.70	981.80	987.30	974.20
GE ₁	1108.38	1161.70	1084.10	1049.90	931.18
GE ₂	860.14	977.63	969.52	989.66	980.44
GE ₂	875.67	967.29	958.58	989.59	979.11
GE ₂	909.27	976.30	951.40	972.70	942.10
HPQ ₁	1000.00	1011.80	1021.60	1094.80	1105.90
HPQ ₁	921.50	996.25	968.78	1072.70	1025.20
HPQ ₁	1015.75	1016.00	1003.80	1026.50	1007.10
HPQ ₂	992.10	971.78	952.37	968.83	946.39
INTC ₁	1005.55	1048.50	984.47	1098.70	985.34
INTC ₁	895.70	1043.30	884.40	1026.40	867.00
INTC ₁	904.78	1036.10	860.96	1041.40	870.00
INTC ₁	942.89	1045.50	911.20	1054.10	870.50
JPM ₁	1000.00	925.70	926.47	1005.80	1010.00
JPM ₁	938.07	932.50	933.60	1006.40	1009.70
JPM ₁	968.19	929.3	928.7	1001.7	997.4
JPM ₁	918.19	871.39	845.57	933.67	901.61

Table 10.2: The pricing error of each model as a percentage of the traded price.

Pricing percentage error.				
Firm ticker	SVSI	SV	SI	Brennan and Schwartz
MCD ₁	-0.0127	0.0306	0.0087	0.0144
MCD ₁	0.0321	0.0726	0.0769	0.0742
MCD ₂	0.1040	0.0738	0.0454	0.0622
MCD ₂	0.1321	0.0982	0.0900	0.0607
AT&T ₁	0.0055	-0.0748	0.0033	-0.0764
CVX ₁	0.0420	0.2021	0.0423	0.2545
CVX ₁	0.0135	0.1432	0.0133	0.2076
CVX ₁	-0.0016	0.0013	-0.0049	0.0054
CVX ₁	-0.0118	0.0128	-0.0169	0.0593
CSCO ₁	-0.0623	-0.0612	-0.0435	-0.0410
CSCO ₁	-0.0629	-0.0648	-0.0593	-0.0628
CSCO ₁	-0.0203	-0.0203	-0.0203	-0.0203
CSCO ₁	0.0300	0.0208	0.0265	0.0129
GE ₁	0.0481	-0.0219	-0.0528	-0.1599
GE ₂	0.1366	0.1272	0.1506	0.1399
GE ₂	0.1046	0.0947	0.1301	0.1181
GE ₂	0.0737	0.0463	0.0698	0.0361
HPQ ₁	0.0118	0.0216	0.0948	0.1059
HPQ ₁	0.0811	0.0513	0.1641	0.1125
HPQ ₁	0.0002	-0.0118	0.0106	-0.0085
HPQ ₂	-0.0205	-0.0400	-0.0235	-0.0461
INTC ₁	0.0427	-0.0210	0.0926	-0.0201
INTC ₁	0.1648	-0.0126	0.1459	-0.0320
INTC ₁	0.1451	-0.0484	0.1510	-0.0384
INTC ₁	0.1088	-0.0336	0.1179	-0.0768
JPM ₁	-0.0743	-0.0735	0.0058	0.0100
JPM ₁	-0.0059	-0.0048	0.0728	0.0764
JPM ₁	-0.0402	-0.0408	0.0346	0.0302
JPM ₁	-0.0510	-0.0791	0.0169	-0.0181
Average	0.0315	0.0134	0.0463	0.0269

Table 10.3: The absolute pricing error of each model as a percentage of the traded price.

Absolute pricing percentage error.				
Firm ticker	SVSI	SV	SI	Brennan and Schwartz
MCD ₁	0.0127	0.0306	0.0087	0.0144
MCD ₁	0.0321	0.0726	0.0769	0.0742
MCD ₂	0.1040	0.0738	0.0454	0.0622
MCD ₂	0.1321	0.0982	0.0900	0.0607
AT&T ₁	0.0055	0.0748	0.0033	0.0764
CVX ₁	0.0420	0.2021	0.0423	0.2545
CVX ₁	0.0135	0.1432	0.0133	0.2076
CVX ₁	0.0016	0.0013	0.0049	0.0054
CVX ₁	0.0118	0.0128	0.0169	0.0593
CSCO ₁	0.0623	0.0612	0.0435	0.0410
CSCO ₁	0.0629	0.0648	0.0593	0.0628
CSCO ₁	0.0203	0.0203	0.0203	0.0203
CSCO ₁	0.0300	0.0208	0.0265	0.0129
GE ₁	0.0481	0.0219	0.0528	0.1599
GE ₂	0.1366	0.1272	0.1506	0.1399
GE ₂	0.1046	0.0947	0.1301	0.1181
GE ₂	0.0737	0.0463	0.0698	0.0361
HPQ ₁	0.0118	0.0216	0.0948	0.1059
HPQ ₁	0.0811	0.0513	0.1641	0.1125
HPQ ₁	0.0002	0.0118	0.0106	0.0085
HPQ ₂	0.0205	0.0400	0.0235	0.0461
INTC ₁	0.0427	0.0210	0.0926	0.0201
INTC ₁	0.1648	0.0126	0.1459	0.0320
INTC ₁	0.1451	0.0484	0.1510	0.0384
INTC ₁	0.1088	0.0336	0.1179	0.0768
JPM ₁	0.0743	0.0735	0.0058	0.0100
JPM ₁	0.0059	0.0048	0.0728	0.0764
JPM ₁	0.0402	0.0408	0.0346	0.0302
JPM ₁	0.0510	0.0791	0.0169	0.0181
Average	0.0566	0.0553	0.0616	0.0683

Table 10.4: The second transaction involving Intel (INTC)

Firm ticker	Traded Price	SVSI	SV	SI	Brennan and Schwartz
INTC	895.70	1043.30	884.40	1026.40	867.00

preferable.

The Brennan and Schwartz pricing model is most likely to be a poor pricing model when the spot level of interest rates or volatility is unusually high/low. However, practitioners in derivatives pricing are likely to use an interest rate that corresponds with the maturity of the instrument, the reason being that such an approach will hopefully take into account any future movement in the spot rate. Assuming that such spot rates prevail throughout the life of the asset would be a mistake, and these are circumstances where the models can deliver some differences in the estimation of price. Looking back at our graph of T-bill rates in the previous section, it can be seen that interest-rates were in the region of 0-5% over the last ten year period. At the peak of T-bill rates in 2006-2007, not allowing for a mean reverting interest rate had a significant impact on pricing. The second transaction involving Intel (INTC), for example, occurred at a time when the T-bill rate was 4.37%. Consequently, we can see a significant discrepancy between those models allowing for stochastic interest rates (SVSI/SI) and those which don't (SV/Brennan and Schwartz). Interestingly, it seems that investors believed this relatively high interest rate to be more than just transitory, as reflected in the traded price. However, there is good reason to believe that incorporating a stochastic interest rate provides a better indication of value. An investor trading at the beginning of 2007 was unlikely to have envisioned the coming recession and the accompanying sharp drop in interest rates. The estimation of our interest rate process incorporates the whole period up until 2009, so our estimates suggest that interest rates revert to fairly low levels in the long term. Investors would not have had the benefit of this perfect foresight.

The lines in the box plot represent the upper quartile, median and lower quartile values. The whiskers extend to the extreme values within $1.5 \times$ the inter-quartile

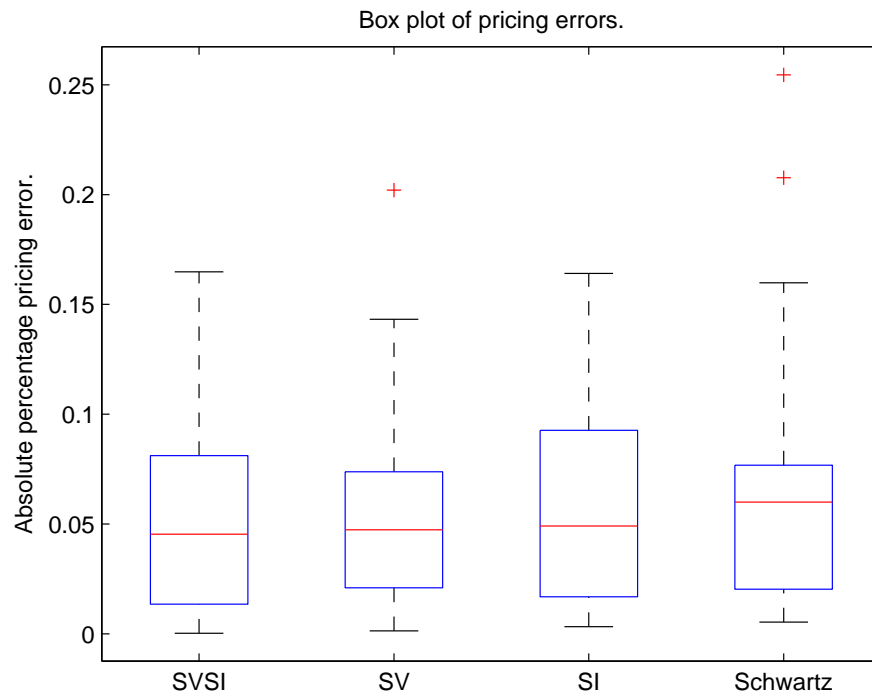


Figure 10.1: Pricing errors for different models.

Table 10.5: Mean squared error for each model

Model	Mean squared error
SVSI	.00541
SV	.00507
SI	.00628
Brennan and Schwartz	.00825

Table 10.6: The third transaction for CSCO

Firm ticker	Traded Price	SVSI	SV	SI	Brennan and Schwartz
CSCO	1033.10	1012.10	1012.10	1012.10	1012.10

range of the box. Outliers are defined as values more than $1.5\times$ the inter-quartile range away from the box. What seems to stand out most is the large outliers associated with Brennan and Schwartz's method. By contrast, the SVSI and SI models have no 'outliers'. Apart from one outlier, the SV model appears to have the least dispersion. Indeed, the SV model has the lowest inter-quartile range of pricing errors and the lowest mean pricing error. This is further reflected by the fact that the SV model has the lowest mean squared error in predicting bond transaction prices, with Brennan and Schwartz being the worst.

As to how the predictive abilities of each model vary with time to maturity, there seems inadequate data to give a concrete conclusion. For the particulars associated with each bond, please refer to appendix D. Most of these bonds have maturities ranging from approximately four to ten years. There is only one bond that has a time to maturity significantly greater than ten years, and this is the INTC bond with a maturity of ten years. Strangely, the SVSI and SI models seem to do fairly poorly in valuing this longer maturity security. The SV and Brennan and Schwartz methods produce far smaller pricing errors for the four transactions associated with this bond.

Another interesting case is the third transaction for CSCO. Our models would suggest that conversion should occur immediately, but the bond still traded at slightly above conversion value. Since immediate conversion was considered optimal, there were no pricing differences between the different models. This brings us to an important point: The models are more similar in their estimation of price the closer we are to maturity, conversion or a bond call; under such circumstances allowing for stochastic volatility or interest rates is of little consequence.

The pricing percentage error was positive on average for all models (See table 10.2). Note that the error is calculated as $\frac{P_m - P_t}{P_t}$, where P_t is the price at which the bond traded and P_m is the estimated price using our model. This tendency of the models to overestimate rather than underestimate the price more on average could be attributable to transaction costs and taxes. Investors are concerned with the payoff they receive net of these costs and hence will price securities accordingly. It is reasonable to expect that any asset valuation model not incorporating taxes and transactions costs will overvalue investments slightly. In the extreme hypothetical case where the tax rate on payoffs is 100%, for example, the worth of any investment would be zero.

In deciding which model to use one should always try to balance accuracy, complexity, available time and circumstances. It may be reasonable to apply a more basic model if we are dealing with a short maturity instrument. Additionally, if the spot levels for interest rates and volatility are approximately at 'normal' levels, then a more basic Brennan and Schwartz model would be a reasonable approximation. However, determining whether the current values of volatility and interest rates are at mean reverting levels necessitates an analysis of these parameters in itself. Consequently, for longer maturity instruments such as bonds, logical reasoning might lead us to apply the more thorough SVSI model. However, from what information we have gathered, our data suggests that the SV model, with a lower mean squared pricing error, is the best model to apply in practice.

Chapter 11

Conclusion.

It seems that the results of Bakshi et al. (1997) are transferrable to convertible bonds. While it appears logically tempting to incorporate stochastic interest rates into our pricing models, our data would suggest that its exclusion would be the best option. Rather, perhaps it is best to use the slightly less complex SV model and treat interest rates as constant. In terms of popularity amongst practitioners, however, it seems that the SVSI model is still favoured in pricing fixed income derivatives. As for any additional areas of research that could assist in making these pricing models more accurate, I think it would be helpful to assess the effect of transaction costs and taxes on the price at which these securities trade, especially since the vast majority of fixed income derivatives are traded over the counter.

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Appendix A

A basic Black-Scholes framework for options pricing.

This short set of notes summarises the more practical aspects of options pricing using a basic Black-Scholes-Merton framework. Firstly, the BSM PDE shall be derived, followed by some possible solutions to this PDE along with supplementary code.

Suppose that the underlying asset, which we'll suppose is a stock, follows GBM such that:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Additionally, suppose that there is an option whose value is dependant on the underlying asset and time; mathematically we can express this as $df = f(S_t, t)$. Using a taylor series expansion, we can then write:

$$df = \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial S}(dS) + \frac{1}{2} \frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}(dS)^2 + \frac{\partial^2 f}{\partial t \partial S}(dt \cdot dS) + \dots$$

Keeping in mind that $dW = \phi \cdot \sqrt{dt}$, where ϕ is a standard normal random variable. Given small values of dt , we'd expect that $(dt)^2$ and $dW \cdot dt = \phi \cdot dt^{1.5}$ are both very small. Also, $E(dW_t^2) = dt$. As such, we can eliminate and substitute as follows:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial S}(dS) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}(dS)^2 \\ &= \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial S}(dS) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}(\mu S dt + \sigma S dW)^2 \\ &= \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial S}(dS) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}(\sigma^2 S^2 dt) \\ &= \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial S}(\mu S dt + \sigma S dW) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}(\sigma^2 S^2 dt) \\ &= \left[\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] dt + \sigma S \frac{\partial f}{\partial S} dW \end{aligned}$$

This is Ito's Lemma. Another important aspect of stochastic calculus is the product rule, derived below. Suppose that we have a function $d(X_t Y_t)$. Applying a Taylor series expansion, we get:

$$\begin{aligned} df &= \frac{\partial f}{\partial X_t}(dX_t) + \frac{\partial f}{\partial Y_t}(dY_t) + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial Y_t^2}(dY_t)^2 + \frac{\partial^2 f}{\partial Y_t \partial X_t}(dY_t dX_t) \\ &= Y_t(dX_t) + X_t(dY_t) + 0 \cdot (dX_t)^2 + 0 \cdot (dY_t)^2 + 1 \cdot (dY_t dX_t) \\ &= Y_t(dX_t) + X_t(dY_t) + (dY_t \cdot dX_t) \end{aligned}$$

A.1 Continuously compounded returns and GBM.

$$\begin{aligned}
 G_t &= \ln(S_t) \\
 dG &= \left[\frac{\partial G}{\partial t} + \mu S \frac{\partial G}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right] dt + \sigma S \frac{\partial G}{\partial S} dW \\
 &= \left[\mu - \frac{1}{2} \sigma^2 \right] dt + \sigma dW
 \end{aligned}$$

Given this result, we can express the distribution of returns under GBM as:

$$\therefore \ln(S_T) - \ln(S_0) \sim \phi \left[\left(\mu - \frac{1}{2} \sigma^2 \right) T, \sigma \sqrt{T} \right]$$

A.2 The Black-Scholes-Merton (BSM) PDE.

Suppose that we create a portfolio consisting of one option and $-\Delta = -\frac{\partial f}{\partial S}$ units of the underlying asset.

$$\begin{aligned}
 \Pi &= f - \Delta S \\
 d\Pi &= df - \Delta dS \\
 &= \left[\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] dt + \sigma S \frac{\partial f}{\partial S} dW - \frac{\partial f}{\partial S} [\mu S dt + \sigma S dW] \\
 &= \left[\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] dt
 \end{aligned}$$

Given that there is no random component to this portfolio return, the portfolio is riskless and therefore must yield the risk-free rate.

$$r\Pi = rf - \frac{\partial f}{\partial S}rS = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right]$$

$$rf = \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

If we incorporate continuous dividends at a rate of q , then the PDE will become:

$$rf = \frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

A.3 Solving the PDE

Since the vast majority of options traded are American in nature, except for a few index options, we often have to use numerical techniques to incorporate the possibility of early exercise.

A closed form solution is available in the pricing of European options, and the formula is as follows:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Where $d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.

Numerical techniques could include:

- Finite difference techniques.
- Binomial tree.
- Trinomial tree.

- Monte Carlo Simulation.

Note, however, that there are several significant assumptions implicit in our derivation of the BSM PDE. Some of the more important assumptions include:

- Lognormally distributed asset returns.
- The path of the underlying asset is continuous.
- Constant volatility.
- Constant interest rates.
- Delta hedging can occur continuously and without cost.

A.4 Finite difference techniques.

Effectively, what finite difference techniques do is approximate the partial derivatives within our PDE using a set of grid points. Given a certain value for the underlying asset at maturity, we will know what the value of our option should be at that time. We can then work through the grid iteratively and solve the PDE at each time step. The result will yield a value for the option today. We shall focus on the Crank-Nicholson (CN) finite difference technique; a method that uses equations from both the implicit and explicit approaches.

$$rf = \frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

Explicit equation:

$$rf_{i,j} = \left[\frac{f_{i,j} - f_{i-1,j}}{\Delta t} \right] + (r - q)S_j \left[\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \right] + \frac{1}{2}\sigma^2 S_j^2 \left[\frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2} \right]$$

Explicit equation shifted up one time period:

$$rf_{i+1,j} = \left[\frac{f_{i+1,j} - f_{i,j}}{\Delta t} \right] + (r-q)S_j \left[\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S} \right] + \frac{1}{2}\sigma^2 S_j^2 \left[\frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{(\Delta S)^2} \right]$$

Implicit equation:

$$rf_{i,j} = \left[\frac{f_{i+1,j} - f_{i,j}}{\Delta t} \right] + (r-q)S_j \left[\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \right] + \frac{1}{2}\sigma^2 S_j^2 \left[\frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2} \right]$$

If we then place an equal weight on the implicit equation and the explicit equation moved up one period, the combined result will be:

$$\begin{aligned} \frac{1}{2} [rf_{i+1,j} + rf_{i,j}] &= \frac{1}{2} \left[\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \right] \\ &+ (r-q)S_j \frac{1}{2} \left[\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S} + \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \right] \\ &+ \frac{1}{2}\sigma^2 S_j^2 \frac{1}{2} \left[\frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{(\Delta S)^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2} \right] \end{aligned}$$

Rearranging this equation will then yield:

$$\begin{aligned} &f_{i,j-1} \frac{1}{2} c_j \Delta t + f_{i,j} \left[1 - \frac{1}{2} b_j \Delta t \right] - f_{i,j+1} \frac{1}{2} a_j \Delta t \\ &= f_{i+1,j-1} \frac{1}{2} c_j \Delta t + f_{i+1,j} \left[1 - \frac{1}{2} b_j \Delta t \right] - f_{i+1,j+1} \frac{1}{2} a_j \Delta t \end{aligned}$$

Where:

- $a_j = \frac{(r-q)S_j}{2\Delta S} + \frac{\sigma^2 S_j^2}{2(\Delta S)^2}$
- $b_j = -r - \frac{\sigma^2 S_j^2}{\Delta S^2}$
- $c_j = \frac{-(r-q)S_j}{2\Delta S} + \frac{\sigma^2 S_j^2}{2(\Delta S)^2}$

This will result in a system of equations $Af_i = Bf_{i+1}$.

$$A = \begin{bmatrix} 1 - \frac{1}{2}b_0\Delta t & -\frac{1}{2}c_0\Delta t & 0 & 0 & 0 \\ -\frac{1}{2}a_1\Delta t & 1 - \frac{1}{2}b_1\Delta t & -\frac{1}{2}c_1\Delta t & 0 & 0 \\ 0 & -\frac{1}{2}a_2\Delta t & 1 - \frac{1}{2}b_2\Delta t & -\frac{1}{2}c_2\Delta t & 0 \\ 0 & 0 & -\frac{1}{2}a_3\Delta t & 1 - \frac{1}{2}b_3\Delta t & -\frac{1}{2}c_3\Delta t \\ 0 & 0 & 0 & -\frac{1}{2}a_4\Delta t & 1 - \frac{1}{2}b_4\Delta t \end{bmatrix}$$

$$B = \begin{bmatrix} 1 + \frac{1}{2}b_0\Delta t & \frac{1}{2}c_0\Delta t & 0 & 0 & 0 \\ \frac{1}{2}a_1\Delta t & 1 + \frac{1}{2}b_1\Delta t & \frac{1}{2}c_1\Delta t & 0 & 0 \\ 0 & \frac{1}{2}a_2\Delta t & 1 + \frac{1}{2}b_2\Delta t & \frac{1}{2}c_2\Delta t & 0 \\ 0 & 0 & \frac{1}{2}a_3\Delta t & 1 + \frac{1}{2}b_3\Delta t & \frac{1}{2}c_3\Delta t \\ 0 & 0 & 0 & \frac{1}{2}a_4\Delta t & 1 + \frac{1}{2}b_4\Delta t \end{bmatrix}$$

What about the boundaries? For very large and small values of S , we shall assume that $\Delta = \frac{\partial f}{\partial S}$ is constant. Consequently, $\Gamma = \frac{\partial^2 f}{\partial S^2} = 0$. The PDE will then be reduced to $rf = \frac{\partial f}{\partial t} + (r - q)S\frac{\partial f}{\partial S}$. As a result, the values of a_j , b_j and c_j will change at the boundaries as follows:

- $a_0 = \frac{(r-q)S_0}{\Delta S}$
- $b_0 = -r - \frac{(r-q)S_0}{\Delta S}$
- $b_N = -r + \frac{(r-q)S_N}{\Delta S}$
- $c_N = -\frac{(r-q)S_N}{\Delta S}$

Dealing with dividends: A known discrete dividend can be dealt with by assuming that the asset price, S_t , has two components, a part that is uncertain, \tilde{S}_t , and a

part that is the PV of the future dividend stream over the options's life. If a cash amount D is paid at time τ , then:

$$\tilde{S}_t = S_t \text{ when } t > \tau$$

$$\tilde{S}_t = S_t - PV(D) \text{ when } t \leq \tau$$

Appendix B

Boundary conditions for the ADI method when solving the stochastic volatility PDE.

Case 1: The lower stock price boundary (S_0):

$f_{sv} = 0$. Therefore $\xi_7 = 0$ and $\xi_{14} = 0$

$f_{ss} = 0$, so take away $\frac{1}{2}V_k S_j^2 \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - 2f_{j,k}^{i+\frac{1}{2}} + f_{j-1,k}^{i+\frac{1}{2}}}{(\Delta S^2)} \right]$

Additionally, we shall have to define $f_s = \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j,k}^{i+\frac{1}{2}}}{(\Delta S)} \right]$, not $f_s = \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{(2\Delta S)} \right]$. So

take away $rS_j \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{(2\Delta S)} \right]$ and add $rS_j \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j,k}^{i+\frac{1}{2}}}{(\Delta S)} \right]$. This results in new definitions for some of the ξ values such that:

- $\xi_7 = \xi_{14} = 0$
- $\xi_1 = \xi_{13} = 0$
- $\xi_3 = \left[\frac{-rS_j}{(\Delta S)} \right]$ and $\xi_{11} = \left[\frac{rS_j}{(\Delta S)} \right]$.
- $\xi_2 = \left[r + \frac{1}{(\Delta\tau/2)} + \frac{rS_j}{(\Delta S)} \right]$ and $\xi_{12} = \left[-r + \frac{1}{(\Delta\tau/2)} - \frac{rS_j}{(\Delta S)} \right]$

Case 2: The upper stock price boundary (S_j):

$f_{sv} = 0$. Therefore $\xi_7 = 0$ and $\xi_{14} = 0$

$f_{ss} = 0$, so take away $\frac{1}{2}V_k S_j^2 \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - 2f_{j,k}^{i+\frac{1}{2}} + f_{j-1,k}^{i+\frac{1}{2}}}{(\Delta S^2)} \right]$

Because we cannot refer to points outside the finite difference grid, we shall redefine $f_s = \left[\frac{f_{j,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{(\Delta S)} \right]$. As a result, take away $rS_j \left[\frac{f_{j+1,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{(2\Delta S)} \right]$ and add $rS_j \left[\frac{f_{j,k}^{i+\frac{1}{2}} - f_{j-1,k}^{i+\frac{1}{2}}}{(\Delta S)} \right]$.

Consequently, some of the lambda values will change as follows:

- $\xi_7 = \xi_{14} = 0$
- $\xi_3 = \xi_{11} = 0$
- $\xi_2 = \left[r + \frac{1}{(\Delta\tau/2)} - \frac{rS_j}{(\Delta S)} \right]$ and $\xi_{12} = \left[-r + \frac{1}{(\Delta\tau/2)} + \frac{rS_j}{(\Delta S)} \right]$.
- $\xi_1 = \left[\frac{rS_j}{(\Delta S)} \right]$ and $\xi_{13} = \left[-\frac{rS_j}{(\Delta S)} \right]$

Case 3: The lower volatility boundary (V_0):

$f_{sv} = 0$. Therefore $\xi_7 = 0$ and $\xi_{14} = 0$

We cannot refer to points outside the boundary, so $\xi_4 = \xi_{10} = 0$.

$f_{vv} = 0$, so you need to take away:

- $\frac{1}{2}V_k \sigma_v^2 \left[\frac{f_{j,k+1}^i - 2f_{j,k}^i + f_{j,k-1}^i}{(\Delta V^2)} \right]$ for the first half-step.
- $\frac{1}{2}V_k \sigma_v^2 \left[\frac{f_{j,k+1}^{i+1} - 2f_{j,k}^{i+1} + f_{j,k-1}^{i+1}}{(\Delta V^2)} \right]$ for the second half-step.

Additionally, we need to redefine f_v such that $f_v = \left[\frac{f_{j,k+1}^i - f_{j,k}^i}{(\Delta V)} \right]$. Therefore, we must take away $(\theta_v - \kappa_{v2} V_k) \left[\frac{f_{j,k+1}^i - f_{j,k-1}^i}{2(\Delta V)} \right]$ and add $(\theta_v - \kappa_{v2} V_k) \left[\frac{f_{j,k+1}^i - f_{j,k}^i}{(\Delta V)} \right]$. At the lower volatility boundary, this will change certain ξ values as follows:

- $\xi_7 = \xi_{14} = 0$
- $\xi_4 = \xi_{10} = 0$
- $\xi_6 = \left[\frac{(\theta_v - \kappa_{v2} V_k)}{(\Delta V)} \right]$ and $\xi_8 = \left[\frac{-(\theta_v - \kappa_{v2} V_k)}{(\Delta V)} \right]$.
- $\xi_5 = \left[\frac{1}{(\Delta\tau/2)} - \frac{(\theta_v - \kappa_{v2} V_k)}{(\Delta V)} \right]$ and $\xi_9 = \left[\frac{1}{(\Delta\tau/2)} + \frac{(\theta_v - \kappa_{v2} V_k)}{(\Delta V)} \right]$.

Case 4: The upper volatility boundary (V_K):

$f_{sv} = 0$. Therefore $\xi_7 = 0$ and $\xi_{14} = 0$.

We cannot refer to points outside the boundary of our finite difference grid, so $\xi_6 = \xi_8 = 0$.

$f_{vv} = 0$, so you need to take away:

- $\frac{1}{2} V_k \sigma_v^2 \left[\frac{f_{j,k+1}^i - 2f_{j,k}^i + f_{j,k-1}^i}{(\Delta V^2)} \right]$ for the first half-step.
- $\frac{1}{2} V_k \sigma_v^2 \left[\frac{f_{j,k+1}^{i+1} - 2f_{j,k}^{i+1} + f_{j,k-1}^{i+1}}{(\Delta V^2)} \right]$ for the second half-step.

Also, we need to redefine f_v such that it does not refer to points outside the boundary. Therefore, $f_v = \left[\frac{f_{j,k}^i - f_{j,k-1}^i}{\Delta V} \right]$. Taking the second half-step as an example, we would have to take away $(\theta_v - \kappa_{v2} V_k) \left[\frac{f_{j,k+1}^i - f_{j,k-1}^i}{2(\Delta V)} \right]$ and add $(\theta_v - \kappa_{v2} V_k) \left[\frac{f_{j,k}^i - f_{j,k-1}^i}{(\Delta V)} \right]$. This changes the ξ values as follows:

- $\xi_7 = \xi_{14} = 0$.

- $\xi_6 = \xi_8 = 0$.
- $\xi_4 = \left[\frac{-(\theta_v - \kappa_v 2V_k)}{(\Delta V)} \right]$ and $\xi_{10} = \left[\frac{(\theta_v - \kappa_v 2V_k)}{(\Delta V)} \right]$.
- $\xi_5 = \left[\frac{1}{(\Delta\tau/2)} + \frac{(\theta_v - \kappa_v 2V_k)}{(\Delta V)} \right]$ and $\xi_9 = \left[\frac{1}{(\Delta\tau/2)} - \frac{(\theta_v - \kappa_v 2V_k)}{(\Delta V)} \right]$.

Appendix C

Boundary conditions for the ADI method when solving the SVSI PDE.

Case 1: At the upper stock price boundary (S_j):

$f_{sv} = 0$. Therefore: $\xi_9 = \xi_{18} = \xi_{27} = 0$.

$f_{ss} = 0$, so take away $\frac{1}{2}V_k S_j^2 \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j-1,k,l}^{i+\frac{1}{3}}}{(\Delta S)^2} \right]$.

We cannot refer to points outside of the finite difference mesh, so we will have

to redefine $f_s = \left[\frac{f_{j,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{\Delta S} \right]$, not $f_s = \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \right]$. Therefore, take away

$r_l S_j \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \right]$, and add $r_l S_j \left[\frac{f_{j,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{\Delta S} \right]$. Consequently, we will get some of the ξ values changing as follows:

- $\xi_9 = \xi_{18} = \xi_{27} = 0$.
- $\xi_1 = \xi_{14} = \xi_{23} = 0$.

Additionally, alter some ξ values such that the new value for each ξ is incremented or decremented to equal $\hat{\xi}$. If we are on multiple boundaries, then a given lambda value may have to be altered several times. For example, if we were on both a lower stock price boundary and an upper interest rate boundary, the final value for ξ_{13} would be $\hat{\xi}_{13} = \xi_{13} + \left[\frac{V_k S_j^2}{(\Delta S)^2} - \frac{r_l S_j}{\Delta S} \right] + \left[\frac{\sigma_r^2 r_l}{(\Delta r)^2} + \frac{(\theta_r - \kappa_r r_l)}{\Delta r} \right]$. Note, however, that the ξ values set equal to zero will always have no value, regardless of whether we are on a boundary for another dimension. ¹

- $\hat{\xi}_2 = \xi_2 + \left[-\frac{V_k S_j^2}{(\Delta S)^2} - \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{13} = \xi_{13} + \left[\frac{V_k S_j^2}{(\Delta S)^2} + \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{22} = \xi_{22} + \left[\frac{V_k S_j^2}{(\Delta S)^2} + \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_3 = \xi_3 + \left[\frac{V_k S_j^2}{2(\Delta S)^2} - \frac{r_l S_j}{2\Delta S} + \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{15} = \xi_{15} + \left[-\frac{V_k S_j^2}{2(\Delta S)^2} + \frac{r_l S_j}{2\Delta S} - \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{24} = \xi_{24} + \left[-\frac{V_k S_j^2}{2(\Delta S)^2} + \frac{r_l S_j}{2\Delta S} - \frac{r_l S_j}{\Delta S} \right]$

Case 2: The lower stock price boundary (S_0):

$f_{sv} = 0$. Therefore: $\xi_9 = \xi_{18} = \xi_{27} = 0$.

$f_{ss} = 0$, so take away $\frac{1}{2} V_k S_j^2 \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - 2f_{j,k,l}^{i+\frac{1}{3}} + f_{j-1,k,l}^{i+\frac{1}{3}}}{(\Delta S)^2} \right]$.

Additionally, we shall have to redefine $f_s = \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j,k,l}^{i+\frac{1}{3}}}{\Delta S} \right]$, not $f_s = \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \right]$.

So take away $r_l S_j \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j-1,k,l}^{i+\frac{1}{3}}}{2\Delta S} \right]$ and add $r_l S_j \left[\frac{f_{j+1,k,l}^{i+\frac{1}{3}} - f_{j,k,l}^{i+\frac{1}{3}}}{\Delta S} \right]$.

This results in some new definitions for ξ values such that:

¹The reason we alter some of the ξ values through increments and decrements rather than providing new definitions is because such an approach will prove easier when writing the computer code to solve this PDE.

- $\xi_9 = \xi_{18} = \xi_{27} = 0.$
- $\xi_3 = \xi_{15} = \xi_{24} = 0.$

Some ξ values will also have to be incremented and decremented as follows:

- $\hat{\xi}_1 = \xi_1 + \left[\frac{V_k S_j^2}{2(\Delta S)^2} + \frac{r_l S_j}{2\Delta S} - \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{14} = \xi_{14} + \left[-\frac{V_k S_j^2}{2(\Delta S)^2} - \frac{r_l S_j}{2\Delta S} + \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{23} = \xi_{23} + \left[-\frac{V_k S_j^2}{2(\Delta S)^2} - \frac{r_l S_j}{2\Delta S} + \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_2 = \xi_2 + \left[-\frac{V_k S_j^2}{(\Delta S)^2} + \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{13} = \xi_{13} + \left[\frac{V_k S_j^2}{(\Delta S)^2} - \frac{r_l S_j}{\Delta S} \right]$
- $\hat{\xi}_{22} = \xi_{22} + \left[\frac{V_k S_j^2}{(\Delta S)^2} - \frac{r_l S_j}{\Delta S} \right]$

Case 3: The upper volatility boundary (V_K):

$f_{sv} = 0$. Therefore: $\xi_9 = \xi_{18} = \xi_{27} = 0$.

$f_{vv} = 0$, so take away $\frac{1}{2}\sigma_v^2 V_k \left[\frac{f_{j,k+1,l}^i - 2f_{j,k,l}^i + f_{j,k-1,l}^i}{(\Delta V)^2} \right]$.

Additionally, we need to redefine $f_v = \left[\frac{f_{j,k,l}^i - f_{j,k-1,l}^i}{\Delta V} \right]$, not $f_v = \left[\frac{f_{j,k+1,l}^i - f_{j,k-1,l}^i}{2\Delta V} \right]$. So we

need to add $(\theta_v - \kappa_{v3} V_k) \left[\frac{f_{j,k,l}^i - f_{j,k-1,l}^i}{\Delta V} \right]$ and take away $(\theta_v - \kappa_{v3} V_k) \left[\frac{f_{j,k+1,l}^i - f_{j,k-1,l}^i}{2\Delta V} \right]$.

The ξ values will then change as follows:

- $\xi_9 = \xi_{18} = \xi_{27} = 0.$
- $\xi_5 = \xi_{10} = \xi_{25} = 0.$
- $\hat{\xi}_4 = \xi_4 + \left[\frac{\sigma_v^2 V_k}{(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_{11} = \xi_{11} + \left[-\frac{\sigma_v^2 V_k}{(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$

- $\hat{\xi}_{22} = \xi_{22} + \left[\frac{\sigma_v^2 V_k}{(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_6 = \xi_6 + \left[-\frac{\sigma_v^2 V_k}{2(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} - \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_{12} = \xi_{12} + \left[\frac{\sigma_v^2 V_k}{2(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} + \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_{26} = \xi_{26} + \left[-\frac{\sigma_v^2 V_k}{2(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} - \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$

Case 4: The lower volatility boundary (V_0)

$f_{sv} = 0$. Therefore: $\xi_9 = \xi_{18} = \xi_{27} = 0$.

$f_{vv} = 0$, so take away $\frac{1}{2}\sigma_v^2 V_k \left[\frac{f_{j,k+1,l}^i - 2f_{j,k,l}^i + f_{j,k-1,l}^i}{(\Delta V)^2} \right]$.

Also, redefine $f_v = \left[\frac{f_{j,k+1,l}^i - f_{j,k,l}^i}{\Delta V} \right]$, not $f_v = \left[\frac{f_{j,k+1,l}^i - f_{j,k-1,l}^i}{2\Delta V} \right]$, such that we do not refer to points outside the grid. Consequently, we'll have to add $(\theta_v - \kappa_{v3} V_k) \left[\frac{f_{j,k+1,l}^i - f_{j,k,l}^i}{\Delta V} \right]$ and take away $(\theta_v - \kappa_{v3} V_k) \left[\frac{f_{j,k+1,l}^i - f_{j,k-1,l}^i}{2\Delta V} \right]$. Our ξ values will then change as follows:

- $\xi_9 = \xi_{18} = \xi_{27} = 0$.
- $\xi_6 = \xi_{12} = \xi_{26} = 0$.
- $\hat{\xi}_5 = \xi_5 + \left[-\frac{\sigma_v^2 V_k}{2(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} + \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_{10} = \xi_{10} + \left[\frac{\sigma_v^2 V_k}{2(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} - \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_{25} = \xi_{25} + \left[-\frac{\sigma_v^2 V_k}{2(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{2\Delta V} + \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_4 = \xi_4 + \left[\frac{\sigma_v^2 V_k}{(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_{11} = \xi_{11} + \left[-\frac{\sigma_v^2 V_k}{(\Delta V)^2} + \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$
- $\hat{\xi}_{22} = \xi_{22} + \left[\frac{\sigma_v^2 V_k}{(\Delta V)^2} - \frac{(\theta_v - \kappa_{v3} V_k)}{\Delta V} \right]$

Case 5: The upper interest rate boundary (r_L):

$f_{rr} = 0$. Therefore, take away $\frac{1}{2}\sigma_r^2 r_l \left[\frac{f_{j,k,l+1}^i - 2f_{j,k,l}^i + f_{j,k,l-1}^i}{(\Delta r)^2} \right]$.

Additionally, we will have to redefine $f_r = \left[\frac{f_{j,k,l}^i - f_{j,k,l-1}^i}{\Delta r} \right]$, not $f_r = \left[\frac{f_{j,k,l+1}^i - f_{j,k,l-1}^i}{2\Delta r} \right]$.

So add $(\theta_r - \kappa_{r3}r_l) \left[\frac{f_{j,k,l}^i - f_{j,k,l-1}^i}{\Delta r} \right]$ and take away $(\theta_r - \kappa_{r3}r_l) \left[\frac{f_{j,k,l+1}^i - f_{j,k,l-1}^i}{2\Delta r} \right]$.

Some of the ξ values will then change as follows:

- $\xi_7 = \xi_{16} = \xi_{19} = 0.$
- $\hat{\xi}_4 = \xi_4 + \left[\frac{\sigma_r^2 r_l}{(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_{13} = \xi_{13} + \left[\frac{\sigma_r^2 r_l}{(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_{20} = \xi_{20} + \left[-\frac{\sigma_r^2 r_l}{(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_8 = \xi_8 + \left[-\frac{\sigma_r^2 r_l}{2(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3}r_l)}{2\Delta r} - \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_{17} = \xi_{17} + \left[-\frac{\sigma_r^2 r_l}{2(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3}r_l)}{2\Delta r} - \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_{21} = \xi_{21} + \left[\frac{\sigma_r^2 r_l}{2(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3}r_l)}{2\Delta r} + \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$

Case 6: The lower interest rate boundary (r_0):

$f_{rr} = 0$. Therefore, take away $\frac{1}{2}\sigma_r^2 r_l \left[\frac{f_{j,k,l+1}^i - 2f_{j,k,l}^i + f_{j,k,l-1}^i}{(\Delta r)^2} \right]$.

Additionally, we will have to redefine $f_r = \left[\frac{f_{j,k,l+1}^i - f_{j,k,l}^i}{\Delta r} \right]$, not $f_r = \left[\frac{f_{j,k,l+1}^i - f_{j,k,l-1}^i}{2\Delta r} \right]$.

So add $(\theta_r - \kappa_{r3}r_l) \left[\frac{f_{j,k,l+1}^i - f_{j,k,l}^i}{\Delta r} \right]$ and take away $(\theta_r - \kappa_{r3}r_l) \left[\frac{f_{j,k,l+1}^i - f_{j,k,l-1}^i}{2\Delta r} \right]$.

Some of the ξ values will be altered as follows:

- $\xi_8 = \xi_{17} = \xi_{21} = 0.$
- $\hat{\xi}_7 = \xi_7 + \left[-\frac{\sigma_r^2 r_l}{2(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3}r_l)}{2\Delta r} + \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_{16} = \xi_{16} + \left[-\frac{\sigma_r^2 r_l}{2(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3}r_l)}{2\Delta r} + \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_{19} = \xi_{19} + \left[\frac{\sigma_r^2 r_l}{2(\Delta r)^2} + \frac{(\theta_r - \kappa_{r3}r_l)}{2\Delta r} - \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_4 = \xi_4 + \left[\frac{\sigma_r^2 r_l}{(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$
- $\hat{\xi}_{13} = \xi_{13} + \left[\frac{\sigma_r^2 r_l}{(\Delta r)^2} - \frac{(\theta_r - \kappa_{r3}r_l)}{\Delta r} \right]$

- $\hat{\xi}_{20} = \xi_{20} + \left[-\frac{\sigma_r^2 r_l}{(\Delta r)^2} + \frac{(\theta_r - \kappa_r 3 r_l)}{\Delta r} \right]$

Appendix D

Characteristics of the bonds used as examples.

Bond Characteristics.				
Bond	Par	Time of transaction	Maturity Date	Shares/bond:
MCD ₁	1000	20-Aug-02	27-Aug-09	36
MCD ₂	1000	28-May-02	28-May-09	6.8528
AT&T ₁	1000	18-Jan-01	20-Jul-06	14.9783
CVX ₁	1000	20-Aug-01	15-Aug-08	18.6566
CSCO ₁	1000	31-Jul-03	15-May-10	48.6088
GE ₁	1000	18-Aug-98	31-Jul-03	11.8248
GE ₂	1000	22-May-02	31-Dec-08	19.9487
HPQ ₁	1000	10-Feb-04	18-Feb-14	29.0416
HPQ ₂	1000	25-Aug-05	31-Mar-09	25.308
INTC ₁	1000	14-Dec-05	15-Dec-35	31.52963
JPM ₁	1000	15-Jan-04	30-Jul-11	11.22

Bond Characteristics.			
Bond	Coupon rate	Coupons PA	Maturity in years
MCD ₁	0	0	7.024657534
MCD ₂	0	0	7.005479452
AT&T ₁	0.01	2	5.504109589
CVX ₁	0.049	2	6.991780822
CSCO ₁	0.025	2	6.794520548
GE ₁	0.15	2	4.953424658
GE ₂	0.0125	2	6.616438356
HPQ ₁	0.03	2	10.03013699
HPQ ₂	0.025	2	3.6
INTC ₁	0.0295	2	30.02191781
JPM ₁	0.03	2	7.542465753

In many of the cases we used more than one bond issue for the same firm as part of our data set. Consequently, we have used a subscript to distinguish different bond issues. For example, MCD₁ and MCD₂ refer to the first and second bond issues from McDonalds respectively. The majority of these bonds are callable, and all of them are convertible. Shares/bond represents the number of shares into which each individual bond is convertible. As for coupons, most of these bonds provide semi-annual payments to bondholders. Additionally, all of the firms, with the exception of Cisco, pay quarterly dividends. Whilst these dividend payments tend to be fairly predictable, implementing the numerical procedure will necessitate an estimation of dividends for the coming year along with an annual growth rate. Callable features will only operate within specific dates during the bonds lifetime, and the call price may vary during specified time periods. All of this information needs to be incorporated into the numerical procedure in order to undertake the pricing process.